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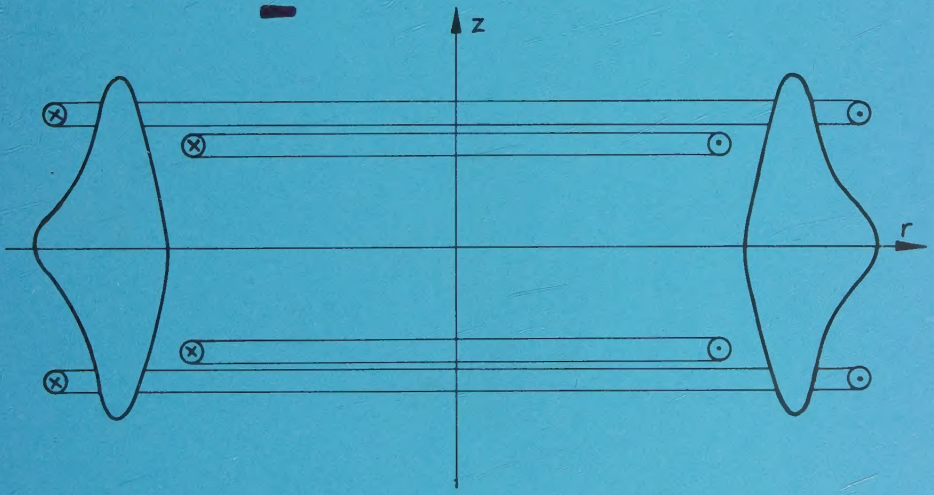
ROLAND RING

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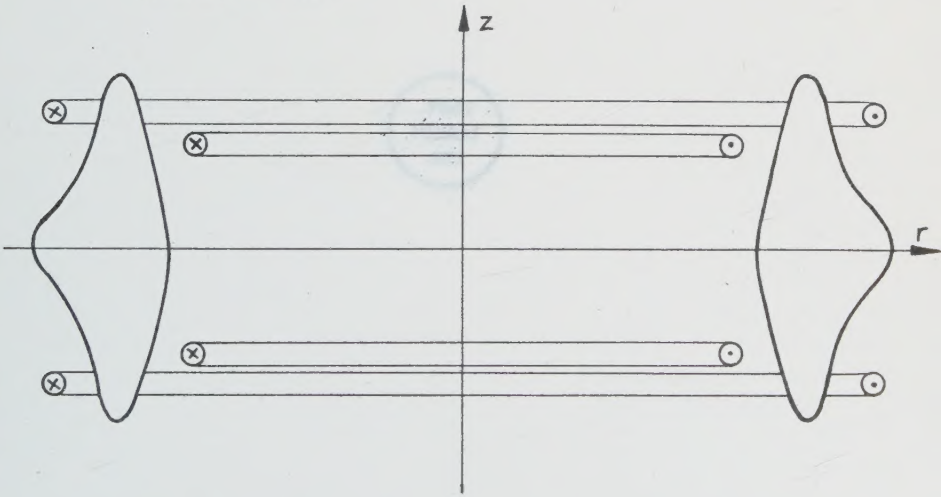


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*To Birgitta and Nils Viktor*

Abstract

A method to compute free boundary equilibria of a toroidal plasma starting from given external sources of the magnetic field is given. The method uses conformal mapping and therefore the torus is modelled by means of two straight columns.
With the magnetic field given at the plasma surface as a boundary-condition an application of the method gives the cross-section of the equilibrium plasma and boundary conditions to the equations valid inside the plasma.
It is shown how the mapping used to calculate the equilibria also simplifies the stability analysis. The method is applied to find Tokamak and Extrap equilibria, and their stability against axisymmetric modes is investigated.

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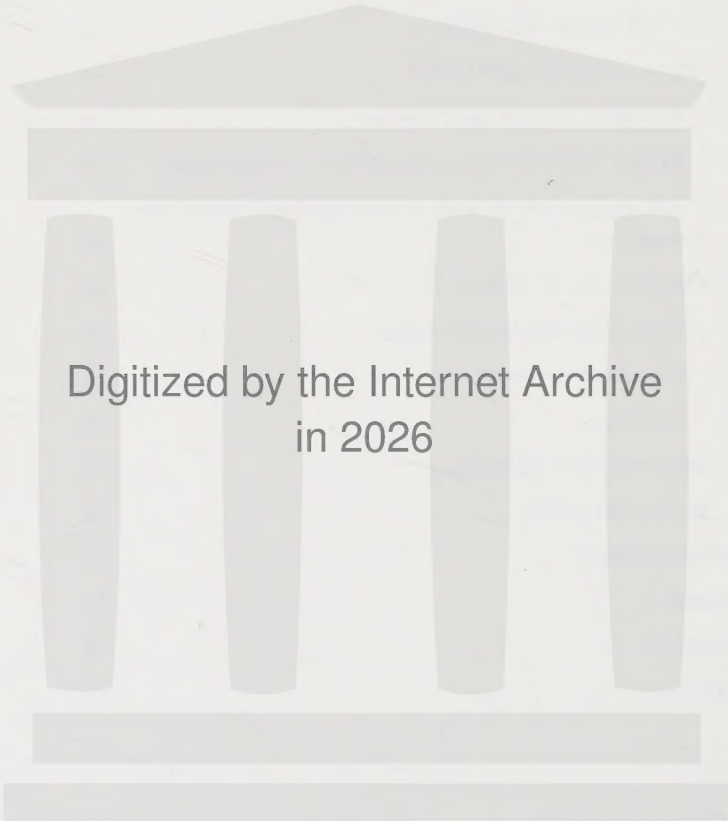
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# 1. INTRODUCTION

The analysis of equilibria and stability of devices for plasma confinement often starts from a given cross-sectional shape of the plasma, usually circular to simplify the analysis. In recent years, with increasing knowledge of plasma behaviour, the interest of investigating plasmas with noncircular cross-sections has increased. The reason is that gains in field economy and stability are expected. An example is the elongation in vertical direction of the cross-section proposed for the Tokamak in the JET programme. In a Tokamak a poloidal magnetic field, often emanating from induced currents in a surrounding conducting shell, holds a toroidal plasma in equilibrium. The field strength that is necessary for the confinement is determined by the pressure of the charged particles of the plasma which in turn for a fully ionized plasma is simply related to the temperature and density. In the fusion regime the temperature and density and thus the pressure are quite high. Studies of the stability show that the Tokamak device needs a strong toroidal magnetic field to maintain stability. The strength of this field depends on the so called safety factor q given by

$$q = \frac{B_\varphi}{B_\theta} \frac{L_\theta}{L_\varphi}$$

where $B_\varphi$ is the toroidal and $B_\theta$ the poloidal magnetic field, $L_\varphi$ the long and $L_\theta$ the short way around the torus. The wellknown Kruskal-Shafranov stability-criterion states that

$$q \geq 2$$

For a torus of circular cross-section the ratio $L_\theta/L_\varphi$ is limited by geometrical conditions. A vertical elongation of the cross-section will increase this ratio and thus decrease the required toroidal field [1]. Recent results, however, have indicated that the picture is not that optimistic as the required safety factor also increases when the cross-section is elongated. Confinement by means of a conducting shell is less efficient for a noncircular than for a circular cross-section, as the maximum confinement time for a shell made of a material with finite conductivity is proportional to the smallest radius

of the cross-section [2], [3]. From this point of view it is advantageous to maintain the equilibrium by currents in suitably situated coils.

A new scheme for plasma confinement has recently been introduced [4], it is called Extrap and consists of a toroidal plasma held in equilibrium by the magnetic field resulting from the currents in four ring-shaped conductors, fig. 1. The field from these conductors is strongly inhomogeneous over the plasma cross-section, especially if the plasma surface is close to the conductors. It is expected that stability of kinkmodes is highly improved by the inhomogeneity of the external field [4], [5]. In the inhomogeneous field of Extrap it is evident that the cross-section of the plasma might differ considerably from a circle.

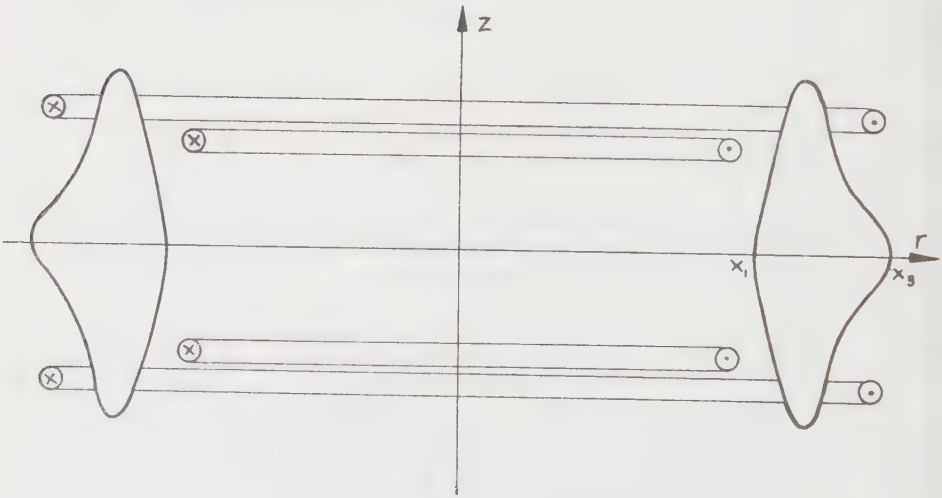



Fig. 1. Extrap. Cut through a plane containing the symmetry axis.

The main efforts of analyzing equilibrium and stability properties of plasma-confinement devices have been devoted to circular plasmas mainly confined by means of a conducting shell. The preceding considerations, however, motivate development of methods by means of which, analyses of free boundary equilibria of plasmas in a magnetic field provided by current carrying coils can be carried out.

One method to handle such situations is the principle of virtual casing [6], where from a given equilibrium it is possible to compute the external currents required to maintain this equilibrium. The way to proceed is to put an ideally conducting shell around the plasma and let the current of the shell fit the boundary conditions of the magnetic field, assuming that there is no effective field outside the shell. Then the field in the plasma emanating from the shell is equal to the true external field while that outside the plasma equals the field from the currents in the plasma but have opposite sign. The total magnetic field is supposed to be known from equilibrium conditions. Thus the field from the external sources can be computed and thereby also the external current distribution. It is often the case that one wants to determine possible equilibria for a plasma under influence of a given external current distribution. Such is for example the case if equilibria of Extrap or of a Tokamak with external fields entirely from current carrying coils are to be determined.

The aim of this work is to present a method by means of which it is possible to find equilibria under such conditions. In section 2 the basic concepts of the method are presented. The presentation is specialized to Tokamaklike and Extraplike configurations in section 3. In section 4 is shown how the stability analysis can be treated by means of the method. In section 5 some equilibria are computed and their stability against axisymmetric modes are tested. In section 6 a final discussion is made.

## 2. AN OUTLINE OF THE METHOD

The basic idea of the method is first to determine a possible magnetic field outside the plasma and at the same time a plasma cross-sectional shape that are possible to maintain with the given external current distribution. When this is done we have boundary conditions at the plasma surface by means of which it is possible to find the magnetic field and the current distribution in the plasma. To make the mathematics more tractable we introduce some simplifications. Firstly the configuration is simplified from that of a torus to two straight noncircular plasma cylinders carrying opposite currents. [7]. The advantage of two cylinders is that the effect of magnetic interaction between different parts of the torus is preserved. The effect of the curving of the torus will, however, get lost. Numerical comparisons of this model with a really toroidal plasma indicate that the interaction between different parts is much more important than the curving and thus that the model is quite useful [7], [8]. We also assume that all quantities depend only of the two space coordinates perpendicular to the cylinders. The analytic simplifications that are provided by these simplifications depend on that they make it possible to represent magnetic potentials, stream functions and magnetic fields with analytic functions wherever the current is zero. We also assume that the plasma can be described by the equations of an ideally conducting noncharged fluid being at rest. We introduce a coordinate system where the r and z coordinates of fig. 1 are replaced by cartesian x and y coordinates. In vacuum the magnetic field can be derived from a potential function $\phi$ as (1)

$$\underline{B} = \nabla\phi$$

and there is a stream function $\phi$ with the property

$\psi$ = constant on magnetic lines of force.

The current $\underline{j}$ , magnetic field $\underline{B}$ and pressure p of the plasma satisfy the equations

$$\underline{j} \times \underline{B} = \nabla p \tag{2}$$

$$\nabla \times \underline{B} = \underline{j} \tag{3}$$

$$\nabla \cdot \underline{B} = 0 \tag{4}$$

$$\left.\begin{array}{l} p > 0 \text{ in the plasma} \\ p = 0 \text{ outside the plasma} \end{array}\right\} \tag{5}$$

The units used have been changed from SI units so that

$$\mu_o \underline{j} \rightarrow \underline{j}$$

$$\mu_o p \rightarrow p$$

We suppose that equation (2) is valid in the whole plasma. A more realistic picture of a fusion plasma would be to adopt equation (2) in the inner region of the plasma and have a partially ionized plasma with finite conductivity and outwardly decreasing degree of ionization as an intermediate link between the fusion plasma domain and the vacuum domain. In this work we suppose that this intermediate region can be ignored. In equation (2) we have also assumed the gravitational force to be weak compared with magnetic and kinetic forces. The pressure is also assumed to be scalar. Symmetry properties of the configuration make it sufficient to study only the first quadrant of fig. 1. The symmetry also, together with equations (2)-(4) give boundary conditions for the magnetic field.

$$B_x = 0 \text{ at the x and y axes} \tag{6}$$

$$B_n = 0 \text{ at the plasma boundary} \tag{7}$$

$$p + \frac{1}{2} B^2 \text{ is continuous across the plasma boundary} \tag{8}$$

$B_n$ is the component of the magnetic field perpendicular to the plasma surface.

If p is discontinuous across the plasma surface then also B is discontinuous according to equation (8). For Amperes law to be fulfilled it is then necessary that there flows a surface current $\underline{K}$ given by

$$\hat{n} \times (\underline{B}_v - \underline{B}_p) = \underline{K} \tag{9}$$

$\hat{n}$ is a unit vector perpendicular to the plasma boundary and pointing into the vacuum region. Subscripts v and p refer to the vacuum and the plasma region respectively. We make the additional simplification that the current of the plasma is flowing perpendicular to the x, y - plane. That is, an eventual magnetic field directed along the cylinder axis is irrotational within the plasma. The magnetic field can be derived from a vector potential $\underline{A}$

$$\underline{B} = \nabla \times \underline{A}$$

with

$$A = \hat{z}\,\psi + \underline{h} \tag{10}$$

and $\underline{h}$ lying in the x, y-plane. Equation (3) gives

$$\nabla^2 \psi = - j \tag{11}$$

and

$$\nabla^2 \underline{h} = 0$$

If further the fields are independent of thez-coordinate equation (4) gives

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

which means that the field parallel to the x, y-plane can be represented entirely by $\psi$. Taking curl of equation (2) gives

$$0 = \nabla \times (\underline{j} \times \underline{B}) = - \nabla \times j \nabla \psi = \nabla j \times \nabla \psi \tag{12}$$

That is j is a function of $\psi$ only. Equation (11) should according to this be written

$$\nabla^2 \psi = -j(\psi) \tag{11a}$$

Outside the plasma j is zero. The current can be eliminated by putting equation (11) into equation (12). The result is

$$\psi_x(\psi_{xx} + \psi_{yy})_y - \psi_y(\psi_{yy} + \psi_{yy})_x = 0 \tag{13}$$

The subscripts denotes differentiation. Outside the plasma it is advantageous to introduce a complex notation. If we write

$$\underline{B} = u\hat{x} - v\hat{y} \tag{14}$$

then

$$w = u + iv \tag{15}$$

is an analytic function of

$$z = x + iy$$

The complex potential

$$f = \phi + i\psi \tag{16}$$

where

$$B = \nabla\phi \tag{1}$$

is also an analytic function of z. The functions w and f are related to each other by

$$w = \frac{df}{dz} \tag{17}$$

The boundary conditions on f follow from equations (6)-(8) and (17). If we choose f(0) = 0 we get

$$\psi(0, y) \equiv 0 \tag{18}$$

$$\phi(x, 0) = 0 \qquad 0 \le x \le x_1 \tag{19}$$

$$\phi(x, 0) = \phi_0 \quad x_3 \leq x < \infty \tag{20}$$

$$\psi(x, y) = \psi_0 \quad \text{at the plasma surface} \tag{21}$$

The constants $\phi_0$ and $\psi_0$ can be computed by means of equation (1), (3) and (16)

$$\phi_0 = \int B \cdot dl = \frac{1}{2} I_p \tag{22}$$

The integration is carried out along the plasma boundary in the first quadrant. $I_p$ is the current of the plasma. $\psi_0$ can be written

$$\psi_0 = \int_0^{x_1} \frac{\partial \psi}{\partial x} dx = - \int_0^{x_1} \frac{\partial \phi}{\partial y} dx = - \int_0^{x_1} \underline{B} \cdot \hat{y} \, dx = - \int B \cdot d\underline{s} \tag{23}$$

The surface of integration has a unit length along the plasma columns and connect the y, z - plane with the plasma surface. That is $\psi_0$ is half the magnetic flux per unit length passing downwards between the two columns.

It is worth noticing that the simplification of only a longitudinal current makes the longitudinal component of the magnetic field passive in the confinement of the plasma as its cross product with $\underline{j}$ becomes zero. But if the plasma is perturbed longitudinally (sausages ) the longitudinal field will straighten the plasma. When we determine the equilibrium the longitudinal field thus can be forgotten but it plays an important role in the stability analysis.

We can now formulate the equilibrium problem more precisely

$$\nabla^2 \psi = j(\psi) \quad \text{in the plasma} \tag{24}$$

or

$$\psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x = 0 \quad \text{in the plasma} \tag{25}$$

$$\nabla^2 \psi = 0 \quad \text{outside the plasma} \tag{26}$$

$$\psi(0, y) \equiv 0 \tag{27}$$

$\psi = \psi_0$ at the plasma boundary (28)

$\psi$ is continuous across the plasma boundary (29)

$$\hat{n}\cdot(\nabla\psi)_v = - B_v \text{ at the plasma boundary} \tag{30}$$

$$(\hat{n}\cdot\nabla\psi)_p = - K - B_v \text{ at the plasma boundary} \tag{31}$$

Boundary condition (29) is arbitrary and only adopted to specify the flux uniquely in the whole space. Outside the plasma $\phi$ is defined wherever j is zero and

$$\nabla^2\phi = 0 \tag{32}$$

$$\phi(x, 0) = 0 \qquad 0 \le x \le x_1 \tag{33}$$

$$\phi(x,0) = \phi_0 \qquad x_3 \le x < \infty \tag{34}$$

provided that the external field has no sources on the axes. To be concrete we will in this work be concerned mostly with Extrap and similar devices. Here f has two singularities in the vacuum region of the first quadrant. These singularities are of the type

$$\lim_{z\to z_k} f(z) = - i \frac{I_k}{2\pi} \ln (z - z_k) \tag{35}$$

where $I_k$ is the current of the conductor situated at $z = z_k$. Transport properties and initial conditions determine the current density in the plasma. So in fact we should solve a much more complicated, timedependent equation that the one we have adopted. Lack of ability besides the fact that the transport properties are not sufficiently known makes it impossible to follow this approach.

Another way to proceed would be to assume a certain current distribution in the plasma and then try to solve eq (24) and (26) in their respective regions of validity together with the boundary conditions (27)-(31). As the position of the boundary is not known a priori this is not an

easy task either. In fact we do not even know if the adopted current distribution is consistent with the outside current distribution so that it allows a solution to exist.

It is in fact no reason to choose to treat the current density in the plasma as known as this quantity has no preference to any other quantity. As the solution of the free boundary problem is considerably simplified by assuming the magnetic field at the plasma boundary to be known,we do that instead. The advantage of this assumption is that we then know boundary conditions for f and w. We also know that they have common singularities. All we thus have to do is to investigate the shape of the domain where f and w simultaneously can be analytic functions under these conditions. This gives the external magnetic field and the position of the boundary. Then we eventually can solve eq (24) or (25) with its, now known, boundary conditions, to get $\underline{j}$ and $\underline{B}$ in the plasma. The magnetic field at the boundary can be given as

$$\frac{1}{|w(s)|} = g(s) \tag{36}$$

or

$$\frac{u(s)}{|w(s)|^2} = g(s) \tag{37}$$

where s is such function that s and the arc-length are singlevalued functions of each other. We now map the vacuum region of the first quadrant of the z-plane conformally on a rectangle in the $\lambda$-plane with the mapping

$$z = z(\lambda) \tag{38}$$

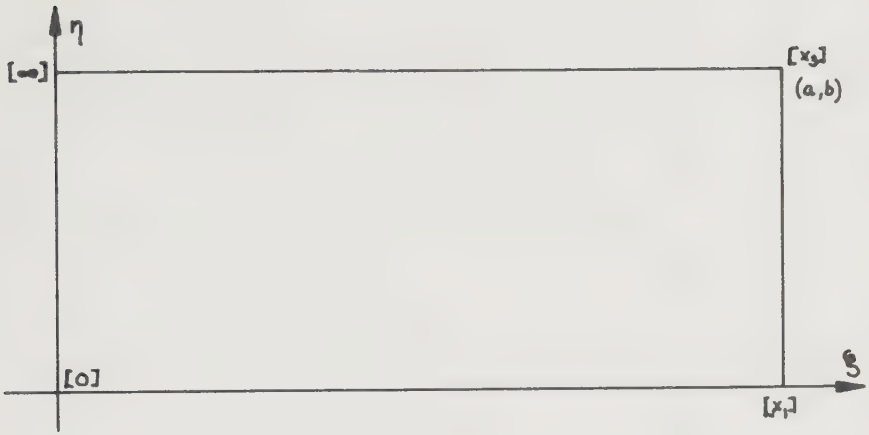



Fig. 2. Map of the vacuum region of the first quadrant of the z-plane. The square brackets contain positions in the z-plane.

with

$$\lambda = \xi + i\eta$$

We carry out the mapping so that infinity maps on ib, the origin at 0 and the plasma surface at the line $\xi = a$, $0 \leq \eta < b$. The external currents are mapped on two points in the interior of the rectangle. As the integral

$$\oint f(z)\, dz$$

transforms to

$$\oint f(\lambda)\, d\lambda$$

the singularities in the rectangle are, according to eq. (32), of the type

$$\lim_{\lambda \to \lambda_k} f(\lambda) = -\frac{I}{2\pi} \ln (\lambda - \lambda_k) \qquad (39)$$

$\lambda_k$ is the image of $z_k$.

The boundary conditions of f in the $\lambda$-plane are according to eq. (27), (28) and (33), (34).

$$\psi(0, \eta) = 0 \tag{40}$$

$$\psi(a, \eta) = \psi_o \tag{41}$$

$$\phi(\xi, 0) = 0 \tag{42}$$

$$\phi(\xi, b) = \phi_o \tag{43}$$

We assume that we know the position of the external currents in the $\lambda$-plane. This has the disadvantage of giving the position of the conductors as a result of the computations but it provides a great simplification. Then it is now possible to compute the complex potential

$$f = f(\lambda) \tag{44}$$

As an extra condition on f we must require that $|W(\eta)|\phi(\eta)$ is monotonically increasing, otherwise the arclength is a multiplevalued function of $\eta$. Eq. (36) and (37) can then be written

$$\frac{1}{|W(\eta)|} = g(\eta) \tag{36a}$$

$$\frac{u(\eta)}{|W(\eta)|^2} = g(\eta) \tag{37a}$$

In eq. (37a) u(a, 0) and u(a, b) must be zero in order to meet the symmetry condition we have assumed. Eq. (37) can also be substituted by the corresponding condition for v. On the plasma boundary $\psi$ is constant that is $\partial\psi/\partial\eta = 0$ there. This together with fig. 2 give the boundary conditions of the mapping.

$$x(0, \eta) = 0 \tag{45}$$

$$y(\xi, 0) = y(\xi, a) = 0 \tag{46}$$

$$\left|\frac{dz}{d\lambda}\right|_{\xi=a} = \frac{\partial\phi}{\partial\eta}\,(a + i\eta)\, g(\eta) \tag{47}$$

if eq. (36a) is used. If we instead use eq. (37a) we get

$$g(\eta) = \frac{u(\eta)}{|w(\eta)|^2} = \frac{\frac{\partial\phi}{\partial\eta}\frac{\partial x}{\partial\eta}}{\left|\frac{dz}{d\lambda}\right|^2} \frac{\left|\frac{dz}{d\lambda}\right|^2}{\left|\frac{\partial\phi}{\partial\eta}\right|^2} = \frac{\frac{\partial x}{\partial\eta}}{\frac{\partial\phi}{\partial\eta}}$$

so that

$$\frac{\partial x}{\partial\eta} = \frac{\partial\phi}{\partial\eta}\,(a + i\eta)\, g(\eta) \tag{48}$$

z must also be singular at the corner $\lambda = ib$ which maps at infinity. To manage the nonlinear boundary condition (47) we write the mapping as

$$\frac{dz}{d\lambda} = e^{R(\lambda)} = e^{r}\,[\cos s + i \sin s] \tag{49}$$

Then $R = r + is$ is an analytic funtion of $\lambda$ fulfilling the boundary conditions

$$s(0, \eta) = 0 \tag{50}$$

$$s(\xi, 0) = 0 \tag{51}$$

$$s(\xi, b) = -\pi \tag{52}$$

$$r(a, \eta) = \ln\left[\frac{\partial\phi}{\partial\eta}\, g(\eta)\right] \tag{53}$$

condition (52) comes from the requirement that $x_1 < x_3$ and that the plasma boundary is perpendicular to the x-axis at $x_1$ and $x_3$. We have also the additional condition

$$\int_0^b \frac{\partial \phi}{\partial \eta} (a + i\eta)\, g(\eta) \cos [s(a + i\eta)]\, d\eta = 0 \qquad (54)$$

that must be satisfied by $s(\lambda)$ in order to make the plasma boundary of the first quadrant to end up at the x-axis.

By applying eq. (36)-(54) the mapping and thereby also the cross-section of the plasma can be determined. Explicitly the point $x_1$ is given by

$$x_1 = \int_0^a \exp (r(\xi, 0))\, d\xi \qquad (55)$$

and the plasma boundary has the coordinates

$$x + i\, y = x_1 + \int_0^\eta \frac{\partial \phi}{\partial \eta'} g\, (-\sin s + i \cos s) d\eta' \qquad (56)$$

The integration is performed along the line $\xi = a$ in the interval $0 < \eta < b$. If we solve the equations

$$\frac{df}{d\lambda} (\lambda_s) = 0$$

and

$$\psi(\lambda) = \psi(\lambda_s)$$

the separatrixes can be found.

The boundary value problem of the plasma region is now completely defined. It consists of solving eq. (24) or (25) under the boundary conditions (29) and (31).

The appearance of a surface current in the equations is due to an approximation where a dense current in a thin but finite shell is described as an infinitely large current density in an infinitely thin shell so that the total current is unchanged. Within this picture the function j can be assumed to be known and the function K will be given as a consequence of boundary condition (31). The physically

most satisfying solution, however, must be one with K equal to zero. Then eq. (24) has too many boundary conditions if j is assumed to be known. The question thus is to determine j so that both boundary conditions (29) and (31) can be satisfied. In this case it seems more convenient to use eq. (25) in place of (24). Eq. (25) is a third order second degree quasi linear equation and it seems likely but is not proved that the boundary conditions (29) and (31) are just enough for it.

We note that eq. (25) is consistent with both eq. (29) and (31) when the current flows perpendicular to the x, y-plane as we have assumed. Integration of eq. (25) over the cross-section gives by means of eq. (12)

$$\int(\psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x)ds = \oint_\Gamma \nabla^2 \psi \nabla \psi \cdot d\underline{l}$$

where $\Gamma$ is the plasma boundary. But with boundary condition (29) the left hand side of this equality must be zero, while it is not influenced at all by eq. (31). We can therefore conclude that (29) and (31) are at least not contradictory as boundary conditions to eq. (25). An especially simple solution of the equations in the plasma is the trivial one with

$$j(\psi) \equiv 0$$

We then get

$$K = - B_v$$

and from eq. (2)-(4) we find that the pressure is constant and the poloidal magnetic field is zero in the plasma. The surface current must then be such that the kinetic pressure in the plasma is exactly balanced by the magnetic pressure outside the plasma so that eq. (8) and (31) are satisfied. As the pressure is constant in the plasma the magnitude of the magnetic field is constant along the plasma boundary. Because of the great analytic simplicity this model offers it will be used in most cases through this work.

## 3. CALCULATIONS OF EQUILIBRIA

In this section elliptic functions are often used. For the convenience of the reader, what is necessary to know about these to be able to follow the presentation is summarized in appendix A. For more details is refered to [9, 10].

The macroscopic parameters of the plasma are its mass M, valume V, temperature T and pressure p which are related to each other by an equation of state

$$F(M, V, p, T) \equiv 0 \tag{57}$$

A given mass of plasma thus varies in volume, pressure and temperature so that equation (57) is satisfied when different external magnetic fields are applied. Besides these parameters we also have the current of the plasma and parameters describing the magnetic field. Which these parameters are, depends on what configuration is being treated.

### 3.1 A Simple Tokamak Equilibrium

In this subsection we show how to compute free boundary equilibria of a Tokamak by use of the surface current model. This case has been treated elsewhere [7] using a different mathematical method. The external currents of this Tokamak provides a homogeneous vertically directed field over the volume of the plasma. We can take the external field to be homogeneous in the whole space which makes the total field homogeneous far from the plasma. The magnetic potential is therefore

$$\phi(x, y) = B_\infty y \tag{58}$$

at large distances from the plasma.

In this case the parameters that describe the external field are

$B_0$ the magnetic field at the origin

$B_\infty$ the magnetic field at infinity

$B_p$ the magnetic field at the plasma boundary

$\psi_0$ half the flux passing between the two columns

L Half the length of the plasma boundary

$\phi_0 = \frac{1}{2} I_p$ half the current of the plasma

Not all of these parameters are independent. The relation between $B_p$, L, and $\phi_0$ is

$$\phi_0 = B_p L$$

Assume that we, at a point Q in the vacuum region with distance $|Q|$ from the origin, have the field $B_Q$. We now ask how to change the scale of the field pattern without distorting its form, or change the field-strength. We multiply the current of each source by a constant k. From eq. (3) with $\bar{B}$ as the mean field along a closed path of length L around a source we get

$$\bar{B} L \lambda = I k$$

That is, if we multiply the current with k we must also expand the lengthscale with k. Alternatively we can keep the currents fixed and change the field by a factor $k^{-1}$. Also then the lengthscale expands with the factor k. The two ways of performing affine mappings correspond to changing the volume of the plasma either keeping the pressure fixed or changing the pressure by the factor $k^{-2}$.

As seen from eq. (22) and (23) $\psi_0$ and $\phi_0$ are proportional to I under such transformations.

The problem can thus be normalized, for example so that we put the magnetic field at the plasma-surface equal to unity and keep the ratio $\psi_0/\phi_0$ and $B_\infty$ to certain values. We can then vary $\phi_0$ until an equilibrium is found. According to eq. (58) the potential must have a singular

point at $\lambda = ib$ in the $\lambda$-plane. An Ansatz that makes the computation of the potential comparatively simple is

$$f(\lambda) = ic\left\{sn\left(\frac{K}{a}\lambda|m\right) - \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi\lambda}{b} - \frac{\phi_o}{bc}\lambda\right\} \tag{59}$$

K is the real quarter period of the elliptic function

m is the parameter of the elliptic function

c is a constant.

Expression (59) fulfils all the boundary conditions for arbitrary coefficients $A_n$ except the condition at the line $\xi = a$. There $sn(\frac{K}{a}\lambda|m)$ assumes the value

$$sn\left[\frac{\lambda}{a}(a + i\eta)|m\right] = nd\left(\frac{K'}{b}\eta\,|m'\right)$$

$m' = 1 - m$

K' is the imaginary quarter period of the elliptic function.

In the following the parameter is omitted in the argument of the elliptic functions. It is understood that if the real quarter period appears in the argument the function has parameter m, if the imaginary quarter period appears the function has parameter m'.

Fourier expansion of nd gives

$$nd\left(\frac{K'}{b}\eta\right) = \frac{\pi}{2\sqrt{m}\,K'}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n\frac{\cos\frac{n\pi\eta}{b}}{\cosh\frac{n\pi a}{b}}\right]$$

we therefore get

$$\psi(a + i\eta) = c\left\{\frac{\pi}{2\sqrt{m}\,K'}\left[1 + 2\Sigma(-1)^n\frac{\cos\frac{n\pi\eta}{b}}{\cosh\frac{n\pi a}{b}}\right]\right.$$

$$\left. - \Sigma A_n \sinh\frac{n\pi a}{b}\cos\frac{n\pi\eta}{b} - \frac{\phi_o a}{bc}\right\}$$

To fulfil boundary condition (41) the coefficients $A_n$ must be

$$A_n = \frac{(-1)^n \pi}{\sqrt{m}\, K' \cosh \frac{n\pi a}{b} \sinh \frac{n\pi a}{b}} \tag{60}$$

The flux function then assumes the value

$$\psi_o = c \left[ \frac{\pi}{2\sqrt{m}\, K'} - \frac{\phi_o a}{cb} \right]$$

at the plasma surface.

The real part of the expression (59) evaluated along the line $\xi = a$ is

$$\phi(a+i\eta) = c \left\{ \frac{\phi_o \eta}{cb} + \Sigma\, A_n \cosh \frac{n\pi a}{b} \sin \frac{n\pi\eta}{b} \right\}$$

and for the derivative of this

$$\frac{\partial \phi}{\partial \eta}(a+i\eta) = \frac{c}{b} \left\{ \frac{\phi_o}{c} + \pi\, \Sigma\, n\, A_n \cosh \frac{n\pi a}{b} \cos \frac{n\pi\eta}{b} \right\} \tag{62}$$

which will be used in boundary condition (53) of the mapping.

The mapping must have a pole at $\lambda = ib$ and satisfy the boundary conditions (50)-(53). This is fulfilled by the function

$$R(\lambda) = \ln\left[ cn\left(\frac{M}{a}\lambda\right) dn\left(\frac{M}{a}\lambda\right) \right] + \sum_{n=o}^{\infty} B_n \cosh \frac{n\pi\eta}{b}$$

M is the real quarter period of the elliptic functions.

M must be chosen such that singularities inside and on the boundary except the double pole at $\lambda = ib$ are avoided. That is

$$\frac{M'}{M} < \frac{b}{d}$$

M' is the imaginary quarter period of the elliptic functions.

For simplicity we let M' approach $\pi/2$ and get

$$R(\lambda) = -2 \ln \left\{\cosh \frac{\eta\lambda}{2b}\right\} + \sum_{n=o}^{\infty} B_n \cosh \frac{n\pi\lambda}{b} \tag{64}$$

The real and imaginary parts of this function are

$$r(\lambda) = -2 \ln\left\{\cosh \frac{\pi\xi}{2b}\right\} - \ln \left\{\cos^2 \frac{\pi\eta}{2b} + \tgh^2 \frac{\pi\xi}{2b} \sin^2 \frac{\pi\eta}{2b}\right\}$$

$$+ \sum_{n=o}^{\infty} B_n \cosh \frac{n\pi\xi}{b} \cos \frac{n\pi\eta}{b} \tag{65}$$

and

$$s(\lambda) = -2 \operatorname{arctg} \left\{\tgh \frac{\pi\xi}{2b} \tg \frac{\pi\eta}{2b}\right\} + \Sigma B_n \sinh \frac{n\pi\xi}{b} \sin \frac{n\pi\eta}{b} \tag{66}$$

To compute the Fourier coefficients $B_n$ we insert the expressions (62) and (65) into the boundary condition (53).

$$\sum_{n=o}^{\infty} B_n \cosh \frac{n\pi a}{b} \cos \frac{n\pi\eta}{b} = 2 \ln \left\{\cosh \frac{\pi a}{2b}\right\}$$

$$+ \ln \frac{c}{b} + \ln \left\{\cos^2 \frac{\pi\eta}{2b} + \tgh^2 \frac{\pi a}{2b} \sin^2 \frac{\pi\eta}{2b}\right\}$$

$$+ \ln \left\{\frac{\phi_o}{c} + \pi \Sigma n A_n \cosh \frac{n\pi a}{b} \cos \frac{n\pi\eta}{b}\right\} \tag{67}$$

We observe that $B_o$ does not enter into equation (66). The ratio c/b will therefore not affect equation (54) which together with equation (67) therefore constitutes a set of equations for the $B_n$'s and $\phi_o/c$. We modify the earlier parametrization so that we keep the ratio a/b fixed instead of $B_\infty$ when we solve the equations. $B_\infty$ can then be computed by means of

$$B_\infty = -\frac{c}{b} \frac{\pi^2}{4\sqrt{m}\, K'} \exp\left[-\Sigma (-1)^n B_n\right]$$

In appendix B we compare our result with that of ref [7] at the limit

$$B_\infty \to 0$$

We see that the results of app B are in agreement with the force balance of two straight parallel conductors carrying opposite currents

$$|B_\infty| - \frac{I}{4\pi x_1} = 0$$

The current corresponding to a unit field at the plasma boundary is

$$I = 2\phi_0$$

i.e.

$$x_1 = \frac{\phi_0}{2\pi|B_\infty|} = \frac{\frac{-1}{|B_\infty|}}{2\ln|B_\infty|}$$

and

$$B_0 = -3\,B_\infty$$

As mentioned before a longitudinal field with the sources entirely outside the plasma can be applied without changing the equilibrium state. As the external poloidal field is homogeneous such a field is necessary to maintain stability. If the poloidal field is sufficiently inhomogeneous certain evidence indicates that the toroidal field is not needed or can be weaker [4, 5], which is advantageous as it allows a larger β-value.

It can be interesting to compare the method to compute the mapping used in ref. 7 to the one used in this work. In ref. 7 the vacuum-region of the z-plane is mapped onto the magnetic field plane with the mapping

$$w = w(z)$$

and to the potential plane through the mapping

$$f = f(z)$$

In the w-plane the vacuum region is mapped to a half-circle as the magnetic field is constant along the plasma boundary and u is zero on both axes. In the f-plane the vacuum region is mapped on a polygon (closed at infinity) as the potential is piecewise constant along the boundary. Thus the interesting regions of the w and f plane can both be mapped onto the upper half of a $\lambda$-plane and in that way equation (38) is decided. The success of this method depends to a high degree on that the images in the z and w planes are simple figures. But if we have singularities in the z-plane we can for example get branch points on the boundary in the w-plane, see fig. 3.

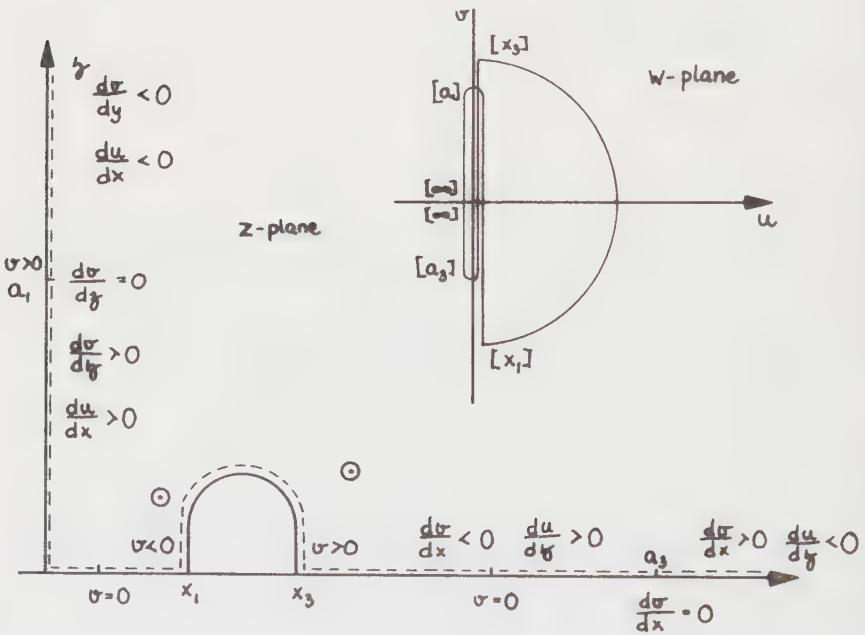



Figure 3. Mapping of the boundary of the vacuum region of the first quadrant of the z-plane onto the w-plane in the presence of two singularities. There are two branchpoints on the boundary in the w-plane. For a more clear illustration the mapping of the dotted line is showed.

The image in the f-plane is not a closed figure either but a periodic function of $\phi$ with the length of the period depending on the external currents. It must also be difficult to get the singularities to coincide so that they constitute a vectorfield of poloidal type. If further the magnetic field at the plasma boundary no longer is constant but varies along the boundary it might be quite difficult to make out such a direct mapping.

### 3.2 Extrap Equilibria in the surface current model

As a second example we treat Extrap-equilibria in the surface current model. This problem differs from the preceding one in that the field and potential have singularities in the vacuum region. With the mapping we have adopted this difficulty is easily resolved. All we have to do is to compute a potential which obeys the boundary conditions (40) - (43) and have the proper number of singularities in the square and insert the result in the boundary condition (67).

To find the potential we cover the $\lambda$-plane with rectangles congruent with the one where the vacuum region is mapped. By repeated mirroring of charges in the sides of the rectangles we can get the potential. It is convenient to arrange the charges so that the potential from them is expressible in elliptic functions. Contrary to what is stated in ref. [9] it is impossible to find such a charge distributions, but we must rest content with that they are satisfied at three sides and adjust the potential to the proper value at the fourth side by means of a Fourier sum. Fig. (4) shows the charge distribution that has been used.

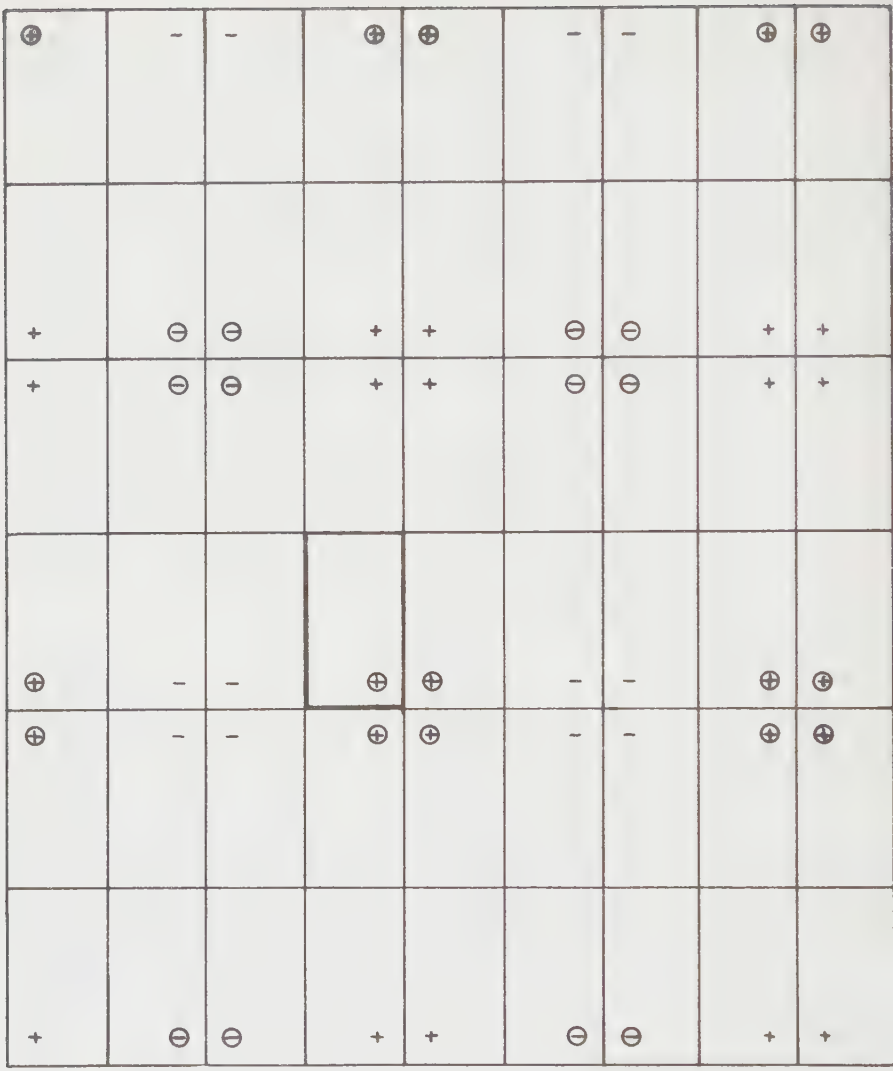


Figure 4. This distribution of point charges satisfies the required boundary conditions on all sides except the vertical left one for the rectangle drawn with bold lines. + signs represent poles, - signs zeros.

From fig. 4 we find that we can use for example ds and cn functions to generate a potential that must be adjusted along $\xi = a$, if we let the distance between poles or zeros be four times the sides of the rectangles in the respective directions. To find the potential from a conductor at $\lambda_{I_k}$, just put a pole of the ds function at $\lambda_K$ and at its mirror points in the other rectangles. In fig. 4 the poles and zeros of the ds functions are encircled and marked with a plus or a minus sign respectively. To keep the potential at three of the sides correct we put zeros of the cn function in the positions $-\lambda_{I_K}$ and its mirror points. The potential from this charge distribution will be

$$f_p^K(\lambda) = -iI_K \ln\left\{m'^{-2} \, ds\left[\frac{K}{2a}(\lambda-\lambda_{I_K})\right] ds\left[\frac{K}{2a}(\lambda-\lambda^*_{I_K})\right]\right.$$

$$ds\left[\frac{K}{2a}(\lambda - \{\lambda_{I_K} + 2(a-\xi_{I_K})\})\right] ds\left[\frac{K}{2a}(\lambda - \{\lambda^*_{I_K} + 2(a-\xi_{I_K})\})\right]$$

$$cn\left[\frac{K}{2a}(\lambda-\lambda_{I_K})\right] cn\left[\frac{K}{2a}(\lambda-\lambda^*_{I_K})\right]$$

$$\left.ds\left[\frac{K}{2a}(\lambda - \{\lambda_{I_K} + 2(a-\xi_{I_K})\})\right] ds\left[\frac{K}{2a}(\lambda - \{\lambda^*_{I_K} + 2(a-\xi_{I_K})\})\right]\right\} \qquad (68)$$

$m'$ is the complementary parameter and the constant $m'^{-2}$ is required to give zero potential at the origin. $f_p^K(\lambda)$ can more simply be written as

$$f_p^K(\lambda) = -iI_K \ln\left\{\frac{ds\left[\frac{K}{2a}(\lambda-\lambda_{I_K})\right]}{ds\left[\frac{K}{2a}(\lambda+\lambda_{I_K})\right]} \frac{ds\left[\frac{K}{2a}(\lambda-\lambda^*_{I_K})\right]}{ds\left[\frac{K}{2a}(\lambda+\lambda^*_{I_K})\right]} \times\right.$$

$$\left.\frac{cn\left[\frac{K}{2a}(\lambda-\lambda_{I_K})\right]}{cn\left[\frac{K}{2a}(\lambda+\lambda_{I_K})\right]} \frac{cn\left[\frac{K}{2a}(\lambda-\lambda^*_{I_K})\right]}{cn\left[\frac{K}{2a}(\lambda+\lambda^*_{I_K})\right]}\right\} \qquad (69)$$

We find that

$$f_p^K(a + in) = f_p^K(a - in) \qquad (70)$$

An expansion in $\cos(n\pi\eta/b)$ will thus be uniformly convergent in the interval $0 < \eta < b$.

To find the potential we make the Ansatz

$$f(\lambda) = ic\Bigg[\sum_{K=1}^{2} G_K \ln\left\{\frac{ds\left[\frac{K}{2a}(\lambda-\lambda_{I_K})\right]}{ds\left[\frac{K}{2a}(\lambda+\lambda_{I_K})\right]}\frac{ds\left[\frac{K}{2a}(\lambda-\lambda_{I_K}^{*})\right]}{ds\left[\frac{K}{2a}(\lambda+\lambda_{I_K}^{*})\right]}\right.$$

$$\left.\frac{cn\left[\frac{K}{2a}(\lambda-\lambda_{I_K})\right]}{cn\left[\frac{K}{2a}(\lambda+\lambda_{I_K})\right]}\frac{cn\left[\frac{K}{2a}(\lambda-\lambda_{I_K}^{*})\right]}{cn\left[\frac{K}{2a}(\lambda+\lambda_{I_K}^{*})\right]}\right\}$$

$$- \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi\lambda}{b} - \frac{\phi_o}{bc}\lambda\Bigg] \tag{71}$$

where $I_K = -cG_K$ According to eq. (41), (70) and (71) we have

$$A_n = \frac{4}{b\sinh\frac{n\pi a}{b}} \int_0^b \sum_{k=1}^{2} G_K \ln\left\{\left|\frac{cn\left\{\frac{K}{2a}(a-\xi_{I_K})+i(\eta-\eta_{I_K})\right\}}{cn\left\{\frac{K}{2a}(a+\xi_{I_K})+i(\eta+\eta_{I_K})\right\}}\right.\right.$$

$$\left.\left.\frac{cn\left\{\frac{K}{2a}\left[(a-\xi_{I_K})+i(\eta+\eta_{I_K})\right]\right\}}{cn\left\{\frac{K}{2a}\left[(a+\xi_{I_K})+i(\eta-\eta_{I_K})\right]\right\}}\right|\right\}\cos\frac{n\pi\eta}{b}\,d\eta \tag{72}$$

The magnetic potential on the line $\xi = a$ is

$$\phi(a+i\eta) = c\sum_{n=1}^{\infty} A_n \cosh\frac{n\pi a}{b}\sin\frac{n\pi\eta}{b} + \frac{\phi_o}{b}\eta$$

which gives

$$\frac{\partial\phi}{\partial\eta}(a+i\eta) = \frac{c}{b}\left\{\frac{\phi_o}{c} + \pi\sum n\,A_n \cosh\frac{n\pi a}{b}\cos\frac{n\pi\eta}{b}\right\} \tag{62}$$

Thus the $A_n$ s found by means of eq. (72) substituted into eq. (67) give the mapping.

3.3 Extrap Equilibria with a Varying Field Along the Boundary

In this subsection we treat Extrap equilibria in connection with the boundary condition (48). The potential is given by expressions (71) and (72). We make the Ansatz

$$z = sn\left(\frac{K}{a}\lambda\right) + \sum_{n=1}^{\infty} D_n \sin\frac{n\pi\lambda}{b} \tag{73}$$

At the line $\xi = a$ we get

$$\frac{\partial x}{\partial \eta} = m' \, Sd\left(\frac{K1}{b}\eta\right) cd\left(\frac{K'}{b}\eta\right) + \frac{\pi}{b} \Sigma \, nD_n \sinh\frac{n\pi a}{b} \sin\frac{n\pi\eta}{b} \tag{74}$$

which is zero at $\eta = 0$ and $\eta = b$ conditions that must be fulfilled for symmetrical reasons. It is also necessary that

$$x(a, 0) < x(a, b) \tag{75}$$

Inserting eq. (74) into (48) we get

$$\frac{\pi}{b} \Sigma \, n \, D_n \sinh\frac{n\pi a}{b} \sin\frac{n\pi a}{b} = \frac{\partial \phi}{\partial \eta} g(\eta) - m' \, sd\left(\frac{K^1}{b}\eta\right) cd\left(\frac{K^1}{b}\eta\right) \tag{76}$$

As $g(\eta)$ is zero at the end points of the interval $0 \leq \eta \leq b$ the Fourier series must converge in the whole interval.

Inserting $D_n$ into eq. (73) and evaluating along $a + i\eta$ gives the plasma boundary. In the plasma the equations to be solved are (24) or (25) with the boundary conditions (28) and (31). $B_v$ is simply related to g and the shape of the plasma surface

$$B_v = \frac{1}{g\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2_\eta}} \tag{77}$$

We are looking for solutions of eq. (25) that have nestled magnetic lines of force. It is reasonable to assume that the current has the same direction in the whole plasma. The current is negative as the magnetic field at the boundary encircles the plasma in the negative direction. This makes the flux to be a sub-harmonic function. The boundary-value problem can be carried from fieldline to fieldline inwards. All the time j is negative and $\psi$ sub-harmonic. That is $\psi$ must be a increasing function of the volume enclosed by the magnetic lines of force. This shows that the solutions of eq. (25) that give nestled magnetic surfaces agree with the natural picture of the solution shown in fig (5), namely that the flux decreases to a minimum value at the magnetic axes when going inwards. What restrictions, however, must be put on $B_v$ to allow such solutions and at the same time make K equal to zero, is not answered in this work.

As the boundary is irregular there is generally no coordinate system in which the boundaries are coordinate lines. It therefore seems probable that equation (25) must be solved numerically.

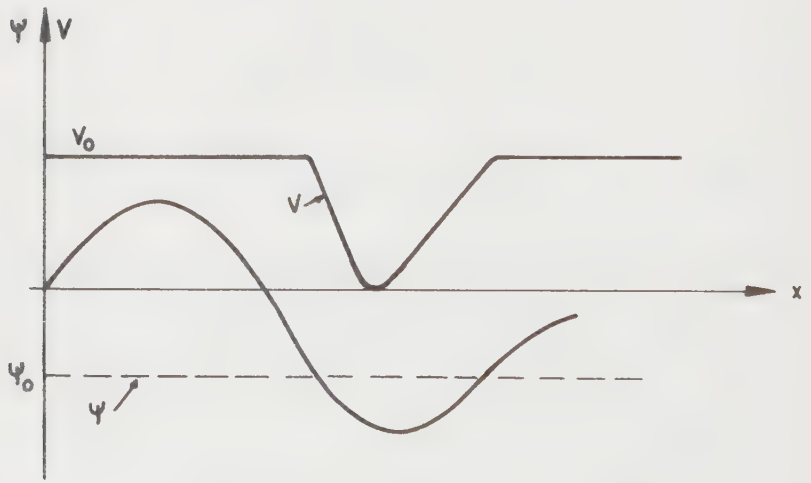



Figure 5. Plot of $\psi$ and V the volume enclosed of the magnetic surface crossing the x-axis. $V_0$ is the volume of the plasma per unit length.

## 4. CALCULATIONS OF STABILITY

### 4.1 Expansions and translations

Stability against expansions of the terms corresponds in the model we have adopted to that the columns strive to reassume their original position when the distance between them is increased by applied external forces.

In this subsection we derive a stability criterion for such perturbations which is valid for a plasma with a circular cross section and with a uniform volume of surface current density. As the plasma under the influence of such perturbations is supposed to move as a rigid body and the current is homogeneous we can without loss of generality study the equilibrium of two thin straight conductors carrying opposite currents and situated in an external field $\underline{B}_e$ which is symmetric with respect to the y,z-plane, see fig. 6. The magnetic field caused by the current in the left conductor is called $B_p$. The equilibrium condition is

$$\underline{B}_e(a, 0) + \underline{B}_p(a, 0) = 0 \tag{78}$$

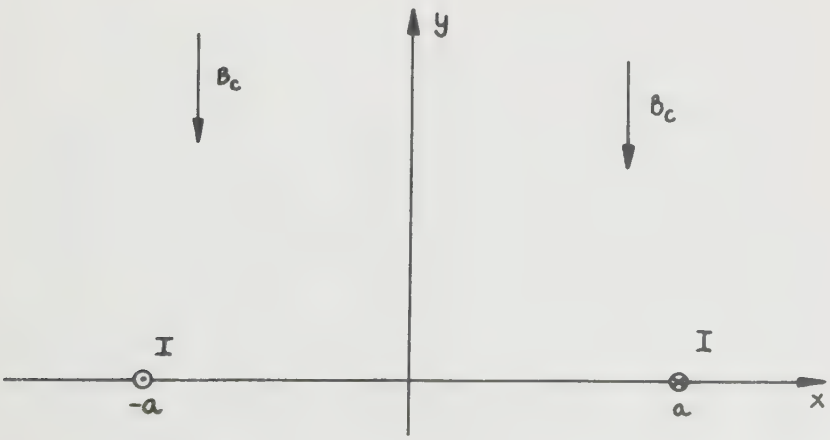



Figure 6. Simplified model used to decide stability criteria against "rigid-body" perturbations.

and as

$$\underline{B}_p(x, 0) = \frac{I}{4\pi x} \hat{y}$$

we have

$$\underline{B}_e(a, 0) = -\frac{I}{4\pi} \frac{\hat{y}}{a} \tag{79}$$

When the distance between the conductors is increased by $\Delta x$ the magnetic field at the right conductor is found to be

$$\underline{B}(a+\Delta x, 0) = \Delta x\left(\frac{\partial B_e}{\partial x} - \frac{I}{4\pi} \frac{\hat{y}}{x^2}\right)_{x=a} = \Delta x\left(\frac{1}{x} \frac{\partial}{\partial x} \times \underline{B}_e\right)_{x=a}$$

by insertion of expression (79). For stability the force acting on the right conductor must be negative for a positive displacement. That is

$$\Delta x \hat{y} \cdot \underline{B}(a + \Delta x, 0) < 0$$

or

$$\frac{\partial}{\partial x}(xB_{ey})_{x=a} < 0 \tag{80}$$

We also use the same scheme to derive conditions for stability against vertical translations. We find that

$$\underline{B}(a, \Delta y) = \Delta y \left.\frac{\partial \underline{B}_e}{\partial y}\right|_{a=0,\ y=0}$$

and

$$\Delta y \hat{x} \cdot \underline{B}(a, \Delta y) > 0$$

so we get

$$\frac{\partial B_e}{\partial y} > 0 \tag{81}$$

The external field has no sources in the plasma, therefore

$$\frac{\partial B_{ex}}{\partial y} = \frac{\partial B_{ey}}{\partial x}$$

For homogeneous and not too strongly inhomogeneous fields condition (80) but not (81) is fulfilled.

For a circular cross-section with homogeneous current distribution or with only a surface current these conditions are valid written as

$$\frac{\partial}{\partial x}(x \bar{B}_e) < 0 \qquad (80a)$$

$$\frac{\partial \bar{B}_{ex}}{\partial y} > 0 \qquad (81a)$$

The bar denotes average over the cross-section or the surface respectively. Expressions (80a) and (81a) agree exactly with the corresponding conditions for a toroidal plasma [11] which can be taken as an indication of that the model of a torus we have adopted when we have computed the equilibria also is useful for stability analysis.

## 4.2 Axisymmetric Modes

The axisymmetric modes correspond in our model to displacements that are independent of the z-coordinate. For a Tokamak with a circular cross-section these modes are stable with a conducting shell around the plasma but only marginally stable without the shell. For an elliptic cross-section they are more unstable but can be stabilized by a conducting shell [12].

For Extrap axisymmetric modes can be expected to be of special importance. To be able to take advantage of the large $\beta$ values Extrap can offer it is necessary to have the conductors as near the plasma as possible. But just the axisymmetric instabilities are then expected to appear and allow the plasma to slip through between the conductors. The axisymmetric modes cannot be stabilized by means of a toroidal field. It is therefore meaningful even for other applications to study them in a configuration with a purely poloidal field [13].

To analyse the stability of axisymmetric modes we will make use of the energy principle in connection with the idealized MHD-equations for adiabatic fluids [14, 15, 16]. We make first a brief review of the derivation of the energy principle.

The idealized MHD-equations are

$$\rho \frac{d\underline{v}}{dt} = - \nabla p + \underline{j} \times \underline{B} \tag{82}$$

$$\frac{d\rho}{dt} + \rho \nabla \cdot \underline{v} = 0 \tag{83}$$

$$\frac{d}{dt} (p\rho^{-\gamma}) = 0 \tag{84}$$

$$E + \underline{v} \times \underline{B} = 0 \tag{85}$$

$$\nabla \times \underline{E} = - \frac{\partial B}{\partial t} \tag{86}$$

$$\nabla \cdot \underline{B} = 0 \tag{87}$$

$$\nabla \times \underline{B} = \underline{j} \tag{88}$$

The boundary conditions for these equations at a fluid-vacuum interface are

$$\hat{n} \times [\underline{E}] = \hat{n} \cdot \underline{v}[\underline{B}] \tag{89}$$

$$\hat{n} \cdot [\underline{B}] = 0 \tag{90}$$

The square brackets in equations (89)-(91) denote the jump of the enclosed quantities across the plasma boundary.

$$\left[p + \frac{1}{2} B^2\right] = 0 \tag{91}$$

We perturb the plasma about an equilibrium described by equations (2)-(5) by means of introducing an infinitely small displacement vector $\underline{\varepsilon}(x, y)$ and linearize the equations with the components of $\underline{\varepsilon}$ as Lagrange-coordinates. For the pressure, magnetic field, and current we get

$$p(\underline{r}, t) = p_o(\underline{r}_o, t) - \gamma p_o \nabla \cdot \underline{\varepsilon} \tag{92}$$

$$\underline{B}_p(\underline{r}, t) = \underline{B}_{op}(\underline{r}_o, t) + \underline{\varepsilon} \cdot \nabla \underline{B}_{op}(\underline{r}_o, t) + \nabla x(\underline{\varepsilon} x \underline{B}_{op}) \tag{93}$$

$$\underline{j}(\underline{r}, t) = \underline{j}_o(\underline{r}, t) - [(\nabla \underline{\varepsilon}) \cdot \nabla] x \underline{B}_{op} + \nabla x[\nabla x(\underline{\varepsilon} x \underline{B}_{op})] + \nabla x[(\underline{\varepsilon} \cdot \nabla) \underline{B}_{op}] \tag{94}$$

subscript "0" refers to unperturbed quantities. Outside the plasma we describe the change in magnetic field caused by the perturbation by means of a vector potential $\underline{A}$ defined by

$$\underline{B}_{1v} = \nabla \times \underline{A} \tag{95}$$

subscript "1" refers to perturbed quantities. Near the boundary if we follow it in its motions we get

$$\underline{B}_v(\underline{r}, t) = \underline{B}_{ov}(\underline{r}_o, t) + \underline{\varepsilon} \cdot \nabla \underline{B}_{ov} + \nabla \times \underline{A} \tag{96}$$

Equation (85) gives

$$\underline{E} = - (\dot{\underline{\varepsilon}} x \underline{B}_{op}) \tag{97}$$

in the plasma, and if we choose $\underline{A}$ to be divergencefree the electric field outside the plasma is

$$\underline{E} = - \dot{\underline{A}}$$

Linearization of the boundary conditions (89)-(91) gives

$$\hat{n} \times \underline{A} = - (\hat{n} \cdot \underline{\xi}) \underline{B}_{ov} \tag{98}$$

and

$$- \gamma p_o \nabla \cdot \underline{\xi} + \underline{B}_{op} \cdot \left[ \underline{\xi} \cdot \nabla \underline{B}_{op} + \underline{\xi} \times (\underline{\xi} \times \underline{B}_{op}) \right] =$$

$$\underline{B}_{ov} \cdot \left[ \underline{\xi} \cdot \nabla \underline{B}_{ov} + \nabla \times \underline{A} \right] \tag{99}$$

By substitution of expressions (92)-(94) into equation (82) we get

$$\rho_o \ddot{\underline{\xi}} = F(\underline{\xi}) \tag{100}$$

with the force density

$$\underline{F}(\underline{\xi}) = \nabla \left[ \gamma p_o \nabla \cdot \underline{\xi} + \underline{\xi} \cdot \nabla p_o \right] + \underline{j}_o \times \left[ \nabla \times (\underline{\xi} \times \underline{B}_{op}) \right]$$

$$+ B_{op} \times \left\{ \nabla \times \left[ \nabla \times (\underline{\xi} \times \underline{B}_{op}) \right] \right\} \tag{101}$$

The force density is thus a function of $\underline{\xi}$ only. That is, we have no dissipation. The sum of change in kinetic and potential energy is therefore zero.

$$\frac{\partial}{\partial t} \left[ \int \frac{1}{2} \rho_o |\dot{\underline{\xi}}|^2 \, dv - \int \underline{\xi} \cdot \underline{F}(\underline{\xi}) dv \right] = 0$$

We carry out the derivation and use equation (100). Then we get

$$\int \underline{\xi} \cdot F(\dot{\underline{\xi}}) dv = \int \dot{\underline{\xi}} \cdot F(\underline{\xi}) dv$$

$\underline{\xi}$ and $\dot{\underline{\xi}}$ can be chosen independently of each other and satisfy both boundary conditions (98) and (99). $\underline{F}$ is thus selfadjoint.

We separate the timevariable from equation (100) by setting

$$\underline{\xi}(\underline{r}_o, t) = \sum_n a_n \underline{\xi}_n(\underline{r}_o) \exp(i\omega_n t)$$

Each of the $\underline{\xi}_n$ are supposed to fulfil boundary conditions (98) and (99), and the equation

$$- \omega_n^2 \rho_o \underline{\xi}_n = F(\underline{\xi}_n) \tag{102}$$

As $\underline{F}$ is selfadjoint the eigenfunctions $\underline{\xi}_n$ can be orthogonalized.

and normalized so that

$$\int \underline{\xi}_n \cdot \underline{\xi}_m \, dv = \delta_{n,m}$$

An arbitrary displacement

$$\underline{\xi} = \Sigma a_n \underline{\xi}_n$$

causes a change in the potential energy according to

$$\delta W = - \frac{1}{2} \int \Sigma \Sigma \underline{\xi}_n F(\xi_m) dv = \frac{1}{2} M \Sigma a_n^2 \omega_n^2$$

It is obvious that at least one negative eigenvalue $\omega_n$ is necessary to make $\delta W$ negative. But it is also sufficient as the corresponding $a_n$ can be arbitrary large compared to other $a_n$'s. Thus for a plasma described by equations (82)-(88) with boundary conditions (89)-(91) a necessary and sufficient condition for stability against small perturbations is that the change in potential energy cannot be made negative by any displacement.

The change of potential energy can be written [16]

$$\delta W = \delta W_F + \delta W_S + \delta W_V \tag{103}$$

with

$$\delta W_F = \frac{1}{2} \int \left\{ |\nabla \times (\underline{\xi} \times \underline{B}_{op})|^2 - \underline{j}_o \times \left[ \nabla \times (\underline{\xi} \times \underline{B}_{op}) \right] \right.$$

$$\left. + r p_o (\nabla \cdot \underline{\xi})^2 + (\nabla \cdot \underline{\xi}) \left[ (\underline{\xi} \cdot \nabla) p \right] \right\} dv \qquad (104)$$

$$\delta W_S = \frac{1}{2} \int (\hat{n} \cdot \underline{\xi})^2 \langle \nabla (p + \frac{1}{2} B_o^2) \rangle \, d\underline{s} \qquad (105)$$

$$\delta W_V = \frac{1}{2} \int (\nabla \times \underline{A})^2 \, dv \qquad (106)$$

The angular brackets in (104) denote the jump of the enclosed quantities across the plasma boundary. The integration in (104) is carried out over the plasma region, in (106) over the vacuum region. Expression (105) is integrated over the plasma surface.

The boundary condition (99) can be abandoned. To every displacement it is namely possible to add an infinitely small displacement that contributes to the energy change with only a infinitely small amount but that has a finite divergence near the boundary. With such an additional displacement it is always possible to satisfy boundary condition (99) without changing anything else [16].

A specialization to a surface current model with only a poloidal magnetic field is now made. The unperturbed field in the plasma is then zero and the pressure is constant. It is obvious that expression (104) then is minimized by a displacement that fulfils

$$\nabla \cdot \underline{\xi} = 0 \qquad (107)$$

which gives $\delta W_F = 0$. As seen from expression (105) only displacements perpendicular to the plasma boundary need to be taken into account. For this case expression (105) can be written

$$\delta W_S = \frac{1}{2} \int \varepsilon^2 \hat{n} \cdot \nabla \frac{1}{2} B_{ov}^2 \, dl \qquad (108)$$

In the following subscripts are suppressed whenever this cannot cause any confusion.

As the magnetic field is tangential to the plasma boundary and the vectorpotential is divergencefree we can write

$$\underline{A} = \hat{z}\psi \tag{109}$$

Expression (106) can by means of partial integration be converted to

$$\delta W_V = \frac{1}{2} \oint \psi \, \hat{n} \cdot \nabla\psi \; d\ell \tag{110}$$

By substitution of expression (109) the boundary condition (98) simplifies to

$$\psi = - B_{ov}\varepsilon \text{ at the plasma boundary} \tag{111}$$

In the vacuum region the flux perturbation must satisfy

$$\nabla^2\psi = 0 \tag{112}$$

We now transform the integrals in the expressions (108)-(112) so that we can take advantage of the conformal mapping we introduced in a preceding section. The mapping was written as

$$\frac{dz}{d\lambda} = e^{R(\lambda)} = e^{r(\lambda)} \, [\cos s(\lambda) + i \sin s(\lambda)] \tag{49}$$

In a surface current model the arc element of the plasma boundary transforms as

$$dl = d\phi = \left(\frac{\partial\phi}{\partial\eta}\right)_{\xi=a} d\eta = e^r \, d\eta \tag{113}$$

according to boundary condition (47). Here and in the following the magnetic field at the plasma boundary is set equal to unity.

After some straighforward vector algebra we get

$$\hat{n} \cdot \nabla \frac{1}{2} B^2 = - B^2 \hat{t} \cdot [(\hat{t} \cdot \nabla)\hat{n}] \tag{114}$$

with

$$\hat{t} \times \hat{n} = \hat{z}$$

From expression (49) we can compute the tangential and normal unit vectors

$$\hat{t} = -\hat{x} \sin s + \hat{y} \cos s \tag{115}$$

$$\hat{n} = -\hat{x} \cos s - \hat{y} \sin s \tag{116}$$

By means of these expressions we find

$$\hat{t} \cdot [(\hat{t} \cdot \nabla)\hat{n}] = \sin s \frac{\partial s}{\partial x} - \cos s \frac{\partial s}{\partial x}$$

From the mapping (49) we conclude that

$$\frac{\partial}{\partial x} = e^{-r}\left(\cos s \frac{\partial}{\partial \xi} - \sin s \frac{\partial}{\partial \eta}\right) \tag{117}$$

$$\frac{\partial}{\partial y} = e^{-r}\left(\sin s \frac{\partial}{\partial s} + \cos s \frac{\partial}{\partial \eta}\right) \tag{118}$$

so we find that

$$\hat{n} \cdot \nabla \frac{1}{2} B^2 = B^2 e^{-r} \frac{\partial s}{\partial \eta} \tag{119}$$

By means of equations (111), (113) and (119) we can express the integral in (108) entirely as an integral along the line $\xi = a$, $0 < \eta < b$ in the $\lambda$-plane.

$$\delta W_s = \int_0^b |\psi|^2 \frac{\partial s}{\partial \eta} d\eta \tag{120}$$

In the same way expression (110) can be rewritten. On the plasma-boundary we get

$$\hat{t} \cdot \nabla \psi = -\cos s \frac{\partial \psi}{\partial x} - \sin s \frac{\partial \psi}{\partial y} = -e^{-r} \frac{\partial \psi}{\partial \xi} \tag{121}$$

To the other sides of the rectangle this quantity transforms as shown in fig. 7. There it is also shown how the line element and the normal vector transform. The expression (107) can thus be written

$$\delta W_v = - \int_{o\eta=o}^{a} \psi \frac{\partial \psi}{\partial \eta} d\xi + \int_{o\xi=a}^{b} \psi \frac{\partial \psi}{\partial \xi} d\eta + \int_{o\eta=b}^{a} \psi \frac{\partial \psi}{\partial \eta} d\xi + \int_{o\xi=o}^{b} \psi \frac{\partial \psi}{\partial \xi} d\eta \qquad (122)$$

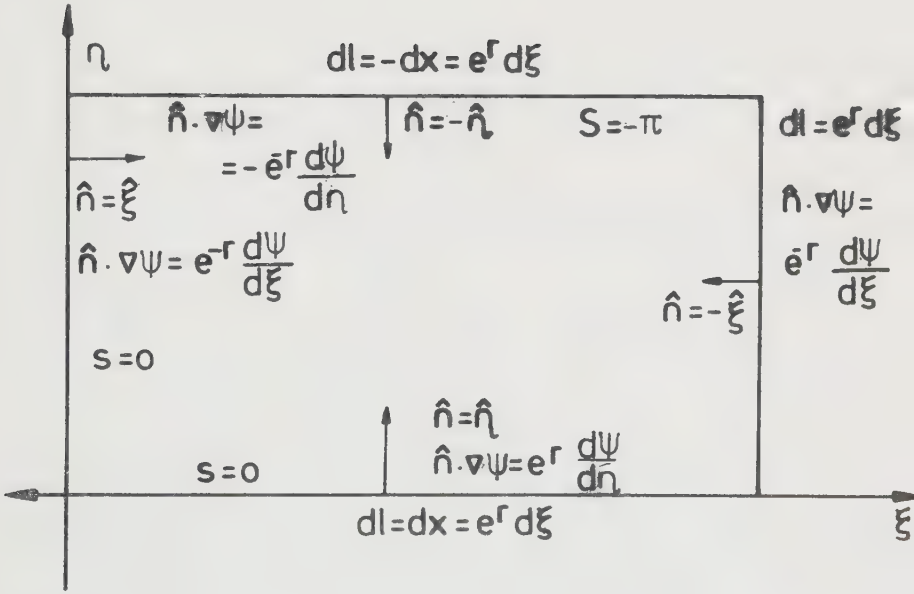



Figure 7. Transformations of line element, normal derivative of flux function and normal unit vector to the boundary in the λ-plane.

To determine the boundary conditions on the y-axis we consider fig. 8, and equation (98) according to which we have

$$\psi = - (\hat{n} \cdot \underline{\varepsilon})\underline{B} \cdot \hat{t}$$

As is shown in fig. 8, $\underline{B} \cdot \hat{t}$ has opposite signs on opposite sides of the y-axis. That is $\psi$ is an odd function of x that requires the boundary condition

$$\psi(o, y) = 0$$

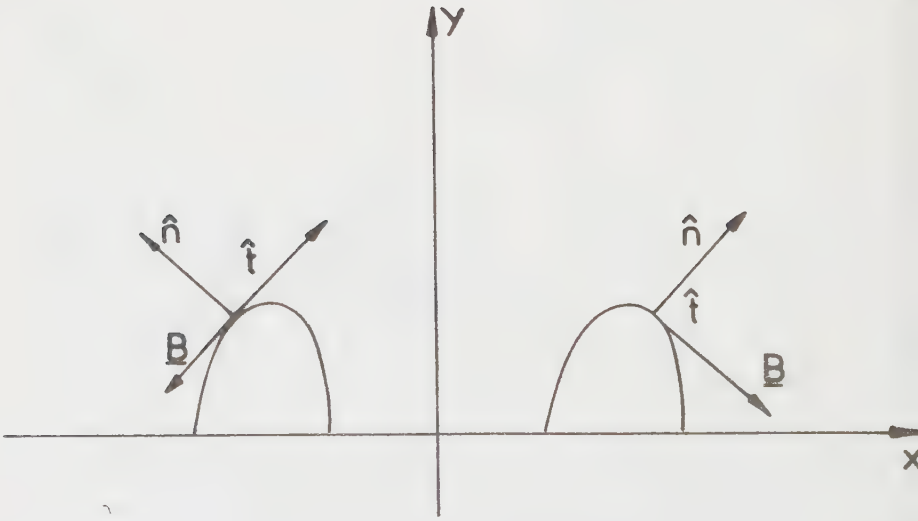



Figure 8. The tangential unit vector and the magnetic field at the plasma boundary have the same direction at the right column but different directions on the left column.

Though we map only the first quadrant of the z-plane we can study arbitrary axisymmetric displacements as it is possible to resolve every displacement in an odd and an even part with respect to the x-axis. This is in view of equation (111) equivalent to an odd or an even flux function with respect to the x-axis. Our problem thus consists of solving equation (112) with boundary-condition (111) on the plasma surface and

$$\psi(o, y) = 0 \tag{123}$$

$$\hat{n} \cdot \nabla\psi(x, o) = 0 \quad \text{for even displacements} \tag{124}$$

$$\psi(x, o) = 0 \qquad \text{for odd displacements} \tag{125}$$

We introduce the perturbation potential $\phi_1$ by the definition

$\phi_1 + i\ \psi_1 = f_1$ is an analytic function.

In the rectangle in the $\lambda$-plane the problem then can be formulated

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0 \tag{126}$$

$$\psi = 0 \quad 0 < \eta < b \qquad \xi = 0 \tag{127}$$

$$\phi = 0 \quad 0 < \xi < a \qquad \eta = 0 \qquad \text{for even displacements} \tag{128}$$

$$\psi = 0 \quad 0 < \xi < a \qquad \eta = 0 \qquad \text{for odd displacements} \tag{129}$$

$$\psi = -\varepsilon \quad \xi = a \qquad 0 < \eta < b \tag{130}$$

$$\phi = 0 \quad 0 < \xi < a \qquad \eta = b \qquad \text{for even displacments} \tag{131}$$

$$\psi = 0 \quad 0 < \xi < a \qquad \eta = b \qquad \text{for odd displacements} \tag{132}$$

The condition that the displacement is divergencefree requires

$$\oint \varepsilon\, dl = 0 \tag{133}$$

with the integration performed along the plasma boundary.

To be sure to satisfy equation (133) we study the stability of modes where the displacement on the boundary is a harmonic function of the arc length.

$$\varepsilon \sim \cos \frac{n\pi\phi}{\phi_o} \ , \ \sin \frac{n\pi\phi}{\phi_o} \tag{134}$$

for the even and odd displacements respectively. The flux function must allow the magnetic field to vanish at $\lambda = ib$. That is

$$\lim_{\lambda \to ib} \left| \frac{df}{d\lambda} \right| e^{-r} = 0$$

From expression (65) we find

$$\exp(-r) = \left( \cosh^2 \frac{\pi\xi}{2b} \cos^2 \frac{\pi\eta}{2b} + \sinh^2 \frac{\pi\eta}{2b} \sin^2 \frac{\pi\eta}{2b} \right)$$

$$\exp \left\{ - \sum_{n=o}^{\infty} B_n \cosh \frac{n\pi\xi}{b} \cos \frac{n\pi\eta}{b} \right\}$$

which vanishes at $\lambda = ib$.

When the complex potential has no singularities we see from the boundary conditions (40)-(43) that we can adopt the displacements

$$\varepsilon(\xi, \eta) = - \sinh \frac{n\pi\psi}{\phi_o} \cos \frac{n\pi\phi}{\phi_o} \tag{135}$$

for even displacements, and

$$\varepsilon(\xi, \eta) = - \sinh \frac{n\pi\psi}{\phi_o} \sin \frac{n\pi\phi}{\phi_o} \tag{136}$$

for odd displacements.

It is simple to compute the total work for simple modes of the type (135)-(136). Either the flux or its normal derivative vanishes at the boundary except for $\xi = a$. From (122) we therefore get

$$\delta W_V = \frac{n\pi}{2} \sinh \frac{n\pi\psi_o}{\phi_o} \cosh \frac{n\pi\psi_o}{\phi_o} \tag{137}$$

independent of the shape of the plasma. For the surface term we get

$$\delta W_S = \sinh^2 \frac{n\pi\psi_o}{\phi_o} \begin{cases} \int_a^b \cos^2 \frac{n\pi\phi}{o} \frac{\partial S}{\partial \eta} d\eta & \text{for even displacements} \\ \int_a^b \sin^2 \frac{n\pi\phi}{\phi_o} \frac{\partial S}{\partial \eta} d\eta & \text{for odd displacements} \end{cases} \tag{138}$$

It is easy to see that $W_S$ always contributes negatively to the total work. Just consider the sum of work for two perturbations one odd and one even. From the boundary conditions (50) and (52) we get.

$$\delta W_S = \sinh^2 \frac{n\pi\psi_o}{\phi_o} \int_o^b \frac{\partial S}{\partial \eta} d\eta = -\pi \sinh^2 \frac{n\pi\psi_o}{\phi_o}$$

If we add (137) the result is

$$\delta W = \delta W_V + \delta W_S = n\pi \sin \frac{n\pi\psi_o}{\phi_o} \cosh \frac{n\pi\psi_o}{o} \left(1 - \frac{1}{n} \operatorname{tgh} \frac{n\pi\psi_o}{\phi_o}\right) \tag{139}$$

As $\operatorname{tgh} \frac{n\pi\psi_o}{\phi_o}$ always is less than unity such a combination is always stable. In appendix B it is shown that this quantity approaches unity if the two columns are moved away from each other. At the same time the cross-section becomes more circular as the interaction between the columns diminish. As seen from expression (139) the stability then tends to be indifferent. This agrees with the result cited in ref.[12] according to which a plasma with circular cross section is marginally stable in absence of a conducting shell.

According to equation (66) we have

$$S(a + i\eta) = -2 \operatorname{arctg} \alpha \operatorname{tg} \frac{\pi\eta}{2b} + \Sigma B_n \sinh \frac{n\pi\xi}{b} \sin \frac{n\pi\eta}{b}$$

with $\alpha = \operatorname{tgh} \frac{\pi a}{2b}$, which gives

$$\frac{\partial S}{\partial \eta}(a + i\eta) = -\frac{\pi}{b} \frac{\alpha}{\cos^2 \frac{2\pi\eta}{2b} + \alpha^2 \sin^2 \frac{2\pi\eta}{2b}} + \frac{\pi}{b} \Sigma\, n\, B_n \sinh \frac{n\pi a}{b} \cos \frac{n\pi\eta}{b} \tag{140}$$

by inserting expression (140) in (138) the total work can be computed

$$\delta W = \frac{n\pi}{2} \sinh \frac{n\pi\psi_o}{\phi_o} \cosh \frac{n\pi\psi_o}{\phi_o} \left[1 + \operatorname{tgh}^2 \frac{n\pi\psi_o}{\phi_o}\right.$$

$$\times \left\{ \begin{array}{ll} \frac{2}{n\pi} \int_o^b \cos^2 \frac{n\pi\phi}{\phi_o} \frac{\partial S}{\partial \eta}\, d\eta & \text{for even modes} \\ \frac{2}{n\pi} \int_o^b \sin^2 \frac{n\pi\phi}{\phi_o} \frac{\partial S}{\partial \eta}\, d\eta & \text{for odd modes} \end{array} \right. \Bigg] \tag{141}$$

For Extrap this method is not possible to use as the equilibrium potential does not satisfy Laplaces equation in the points where the conductors are situated. Instead we adopt the expansion

$$\psi_e^m(\xi, \eta) = \frac{E_o^m \xi}{2a} + \sum_{n=1}^{\infty} E_n^m \sinh \frac{n\pi\xi}{b} \cos \frac{n\pi\eta}{b} \tag{142}$$

for even and

$$\psi_o^m(\xi, \eta) = \sum_{n=1}^{n} D_n^m \sinh \frac{n\pi\xi}{b} \sin \frac{n\pi\eta}{b} \tag{143}$$

for odd modes. $E_n^m$ and $D_n^m$ are found by

$$E_n^m \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_o^b \cos \frac{m\pi\phi}{\phi_o} \cos \frac{n\pi\eta}{b}\, d\eta \tag{144}$$

$$D_n^m \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_o^b \sin \frac{m\pi\phi}{\phi_o} \sin \frac{n\pi\eta}{b} \, d\eta \tag{145}$$

By applying Parseval's identity we then get

$$\delta W_v = \frac{b}{2} \left[ \frac{(E_o^m)^2}{4a} + \frac{\pi}{b} \Sigma \, n(E_n^m)^2 \cosh \frac{n\pi a}{b} \sinh \frac{n\pi a}{b} \right] \tag{146}$$

for even and

$$\delta W_v = \frac{\pi}{2} \Sigma \, n(D_n^m)^2 \cosh \frac{n\pi a}{b} \sinh \frac{n\pi a}{b} \tag{147}$$

for odd displacements.

The contribution from the surface is

$$\delta W_S = \begin{cases} \int_a^b \cos^2 \frac{m\pi\phi}{\phi_o} \frac{\partial S}{\partial \eta} \, d\eta & \text{for even displacements} \\ \int_o^b \sin^2 \frac{m\pi\phi}{\phi_o} \frac{\partial S}{\partial \eta} \, d\eta & \text{for odd displacements} \end{cases} \tag{148}$$

It is immediately clear that also the displacements

$$\varepsilon \sim \sin \frac{n\pi\eta}{b} \tag{149}$$

satisfy condition (133) as they do not change the volume. If that also is the case with

$$\varepsilon \sim \cos \frac{n\pi\eta}{b} \tag{150}$$

must, however, be closer examined. According to boundary condition (111) we have

$$\oint \varepsilon \, dl = - \oint \psi \, dl = -2 \sinh \frac{n\pi a}{b} \int_0^b \cos \frac{n\pi\eta}{b} \frac{\partial \phi}{\partial \eta} \, d\eta =$$

$$= -2 \sinh \frac{n\pi a}{b} \left\{ \left[ \frac{b}{n\pi} \sin \frac{n\pi\eta}{b} \right]_0^b - \frac{b}{n\pi} \int_0^b \sinh \frac{n\pi r}{b} \frac{\partial^2 \phi}{\partial r^2} \, dr \right\}$$

the first term is zero. The integral can be computed by means of equation (62). We get

$$\oint \psi \, dl = \frac{c\pi}{b} n A_n \sinh \frac{n\pi a}{b} \cos \frac{n\pi a}{b}$$

which generally is different from zero. The modes (150) can thus not be used directly for expansion of even displacements but these must be expanded in functions of type (142) appropreately normalized. The odd modes (149) can directly be used as a functionsbase for expansion of arbitrary odd displacements. In the preceding subsection we concluded that of the two types of perturbations studied there the vertical displacement was the most unstable of them for a circular cross-section and the same can be expected to remain true for a vertically elongated cross-section. As vertical translations correspond to first order odd and expansions to first order even axisymmetric perturbation we can expect the odd perturbations to be the most critical.

We compute the stability of simple modes of type (149). The solution to Laplace's equation satisfying boundary conditions (111), (129) and (130) is

$$\psi_1 = \sinh \frac{n\pi\xi}{b} \sin \frac{n\pi\eta}{b} \tag{151}$$

In expression (121) all terms except the integral along $\xi = a$ still disappear. We get

$$\delta W_v = \frac{n\pi}{2} \sinh \frac{n\pi a}{b} \cosh \frac{n\pi a}{b} \tag{152}$$

Independently of the shape of the plasma. From expression (119) we get

$$\delta W_S = \sinh^2 \frac{n\pi a}{b} \int_a^b \sin^2 \frac{n\pi\eta}{b} \frac{\partial S}{\partial \eta} \, d\eta \tag{153}$$

We introduce the function

$$I(n, \alpha) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin^2 2nx}{\cos^2 x + \alpha^2 \sin^2 x} \, dx \tag{154}$$

and compute the contribution from the Fourier series of expression (140)

$$\int_0^b \sin^2 \frac{n\pi\eta}{b} \cos \frac{m\pi\eta}{b} \, d\eta = -\frac{b}{4} \delta_{2n,m} \tag{155}$$

For the total work we then get

$$\delta W = \frac{n\pi}{2} \sinh \frac{n\pi a}{b} \cosh \frac{n\pi a}{b} \left[ 1 - \frac{2\alpha \, \alpha_{2n}}{n} I(n, \alpha) \right.$$

$$\left. - \frac{2\alpha^2_{2n}}{1+\alpha^2_{2n}} B_{2n} \cosh \frac{2n\pi a}{b} \right] \tag{156}$$

with

$$\alpha_{2n} = \operatorname{tgh} \frac{n\pi a}{b} \tag{157}$$

Also when the unperturbed magnetic field is not constant on the plasma surface the preceding equations are valid and useful if boundary condition (47) is used. The magnetic field $\underline{B}_{ov}$ must of course be reinserted where it has been assumed to be unity. For example expression (113) changes to

$$dl = \frac{1}{|W|} d\phi = g(\eta) \left( \frac{\partial \phi}{\partial \xi} \right)_{\xi=a} d\eta = e^r d\eta$$

A comparison between the conditions (47) and (48) shows that, if (48) is used to determine the equilibrium, the substitutions to make are

$$\sin s = -\frac{u}{|W|}$$

$$\cos s = \frac{v}{|W|}$$

$$e^r = \frac{\partial \phi}{\partial n} \frac{1}{|W|}$$

## 4.3 Kinkmodes

We give the perturbing displacement a periodic dependence of the z-coordinate with wavenumber k. It is found that the mapping (49) (or eventually (73)) also then can be used so we can solve the equation of the vacuum region in a domain with a regular boundary. This time, however, at the expense of a more complicated equation. The kinkmodes can be stabilized by a longitudinal magnetic field. Such a field is therefore included here. We take the longitudinal field to be constant on both sides of the y, z-plane but it has opposite directions in the two halves of the space. Such a field can be generated by a vertical surface current in the y, z-plane. As the y, z-plane contains magnetic lines of force of the poloidal field, such a current can be included without changing the equilibrium.

The perturbation depends on the z-coordinate so that the vector potential will not get the simple form it has in expression (109). We will therefore reformulate the boundary condition (98) in order to make use of the scalar potential instead of the vector potential.

If the displacement has no z-component we find from expression (98)

$$A_z\hat{z} = -(\hat{n}\cdot\underline{\varepsilon})B_t\hat{z} = \underline{\varepsilon} \times \underline{B}_t$$

and if the tangential component of the displacement also is zero.

$$A_t\hat{t} = (\hat{n}\cdot\underline{\varepsilon})B_z\hat{t} = \underline{\varepsilon} \times B_z\hat{z}$$

The subscripted B's refer to respective components of $\underline{B}_{ov}$.

We thus have

$$\underline{A} = \underline{\varepsilon} \times \underline{B}_{ov} - A_n\hat{n}$$

But

$$\hat{n}\cdot\{\nabla x(A_n\hat{n})\} = \nabla\cdot(A_n\hat{n} \times \hat{n}) + A_n\hat{n}\cdot(\nabla \times \hat{n}) = 0$$

So that

$$\hat{n}\cdot(\nabla \times \underline{A}) = \hat{n}\cdot\left[\nabla x(\underline{\varepsilon} \times \underline{B}_{ov})\right]$$

The right hand side of this expression can also be expressed in terms of the scalar potential. We then get

$$\hat{n}\cdot\nabla\phi_1 = \hat{n}\cdot\left[\nabla x(\underline{\varepsilon} \times \underline{B}_{ov})\right] \tag{158}$$

$W_F$ is still minimized by a divergencefree displacement. Expression (158) can therefore be written

$$\hat{n}\cdot\nabla\phi_1 = B\cdot\nabla\varepsilon + \varepsilon\underline{B}\cdot\left[(\hat{n}\cdot\nabla)\hat{n}\right] + \hat{n}\left[(B\cdot\nabla)\hat{n}\right]$$

which can easily be expressed in the parameters of the mapping.

$$\frac{\partial\phi}{\partial\xi} = B_t\left(\frac{\partial\varepsilon}{\partial\eta} + \varepsilon\frac{\partial s}{\partial\xi}\right) + B_z e^r \frac{\partial\varepsilon}{\partial z} \tag{159}$$

For kinkmodes with wavenumber k equation (126) transforms to the $\lambda$-plane as

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} - k^2 e^{2r}\phi = 0 \tag{160}$$

$\phi$ is an even function when  is odd and vice versa, the boundary conditions of $\phi_1$ in the $\lambda$-plane are

$$\frac{\partial\phi_1}{\partial\eta} = 0 \quad 0 < \eta < b \quad \xi = 0 \tag{161}$$

$$\frac{\partial \phi_1}{\partial \eta} = 0 \begin{cases} 0 < \xi < a & \eta = 0 \\ 0 < \xi < a & \eta = b \end{cases} \quad \text{for odd displacements} \tag{162}$$

(159) and (128), (131) for even displacements.

For kinkmodes expressions (120) and (122) must be replaced by

$$\delta W_s = \int_0^b |\varepsilon| \frac{\partial s}{\partial \eta} \, d\eta \tag{163}$$

$$\delta W_v = \int_0^b \int_{\xi=a} \phi \frac{\partial \phi}{\partial \xi} \, d\eta \tag{164}$$

The contribution $\delta W_F$ can of course not be computed by means of the mapping but must be treated directly in the z-plane or eventually transformed to a plane where the plasma cross-section maps onto a circle. As the pressure still is constant and the current zero in the plasma expression (104) is

$$\delta W_F = \frac{1}{2} \int |\nabla \times (\underline{\varepsilon} \times B_{op})|^2 \, dV \tag{165}$$

This expression is minimized if the integrand satisfies [16]

$$\nabla \times \left[ \nabla \times (\underline{\varepsilon} \times B_{op}) \right] = 0$$

Expression (165) can therefore be written

$$\delta W_F = \frac{1}{2} \int |\nabla \phi_p|^2 \, dV = 0 \tag{166}$$

where the potential $\phi_p$ satisfies

$$\nabla^2 \phi_p = 0 \tag{167}$$

By comparing (165) and (167) we find that the boundary condition for equation (167) is

$$\hat{n} \cdot \nabla \phi_p = B_z \frac{\partial \epsilon}{\partial z} \tag{168}$$

Expression (166) can of course by means of equation (167) be integrated by parts so that

$$\delta W_F = \frac{1}{4} \oint \phi_p \, \hat{n} \cdot \nabla \phi_p \, d\ell \tag{169}$$

where the integral is carried out along the plasma boundary. The extra factor 1/2 comes from that $\delta W_s$ and $\delta W_v$ in (163) and (164) represent only half the energychange. To arrive at (169) we applied (166) to a suitable length of the plasma columns so that the contributions from the end points cancelled.

The stability of kinkmodes is not meaningful to study in a surface current model with only a poloidal field and as our main interest is in Extrap a treatment of kinkmodes with a poloidal field is not carried out here. It would, however, be very interesting to study the influence of an inhomogenous internal poloidal field on the kinkmode-stability. It is also our hope to get an opportunity to perform such a study as the method presented here is very convenient to compute the equilibrium and potential energy change outside the plasma.

## 5. SOME COMPUTED EXAMPLES

Equations (54) and (67) have been solved numerically for some cases of interest. As potential functions both expressions (59) and (71) and a linear combination of these have been used. A few of the computed equilibria are shown in fig 9-28. The currents $I_{e1}$, $I_{e2}$ refer to the sum of the current in the first quadrant and its mirror current in the fourth quadrant.

As the aim of this work more is to provide a method for computation of equilibrium cross-sections of plasmas and for investigating their stability, rather than to carry out the investigation, and as the equations have many parameters especially if we take expression (71) as potential function, which is the case that seems most interesting to us, no systematic examination of how equilibria and their stability depend on the parameters has been made. The computations are only carried out to test the efficiency of the method, and eventually to get a grasp of how the plasma is acting in the presence of the current-carrying conductors.

We have used the surface current model which we know is unstable for perturbations that are localized enough [16]. That this also is the case for axisymmetric perturbations can be realized if we let $\varepsilon(\eta)$ be a displacement that is different from zero in an subinterval $\Delta\eta$ of the interval $0 < \eta < b$ and zero outside that subinterval. We choose $\Delta\eta$ such that $\partial s/\partial\eta$ is negative in it. The boundary conditions for the flux function are (127)-(132). The displacement must of course be chosen such that expression (133) is fulfilled. The required boundary conditions are satisfied if we take the flux function to be

$$\psi(\varepsilon,\eta) = - \varepsilon(\eta) \sinh k \xi$$

where k is an arbitrary constant. For the change in potential energy we then get

$$\delta W = k \int \varepsilon^2(\eta) \sinh ka \cosh ka \, d\eta - \int \varepsilon^2(\eta) \sinh ka \left|\frac{\partial s}{\partial \eta}\right| d\eta =$$

$$= \Delta\eta \sin ka \left\{ k \varepsilon^2(\eta_1) \cosh ka - \varepsilon^2(\eta_2) \sinh ka \left| \frac{\partial s}{\partial \eta} \right| \right\}$$

But according to the boundary-condition (52) the mean value of $\partial s/\partial \eta$ is $-\pi/b$. So therefore it must be some interval where $\partial s/\partial \eta$ is negative and

$$\left| \frac{\partial s}{\partial \eta} \right| \geq \frac{\pi}{b}$$

so by choosing

$$k < \frac{\pi}{b} \cot ka \quad \frac{\varepsilon^2(\eta_2)}{\varepsilon^2(\eta_1)}$$

it is always possible to obtain a negative change of the total potential energy as long as the magnetic field inside the plasma is zero. That the flux function does not satisfy equation (162) is not a serious objection against the preceding discussion. In subsection 4.3 we have already used the fact that a solution of (126) minimizes $\delta W_v$ under the boundary-conditions (127)-(132). So if the flux function we have chosen makes $\delta W_v$ negative we certainly find a function fulfilling (126) that also makes $\delta W_v$ negative.

We therefore choose to study simple modes having the form of expression (142) or (143). It is easily seen from expression (156) that the mode $m = 1$ is the most critical one for odd perturbations. Some careful consideration of expressions (146) and (148) shows that it also is so for even modes in most cases. In the expansion (142) namely, the term $n = m$ is dominating if we look at equilibria the potential of which is not too far from linear at the side $\xi = a$ of the rectangle in the $\lambda$-plane. At the same time the Fourier coefficients of $\partial s/\partial \eta$ are usually decreasing, and as the contribution of the Fourier expansion of the mapping to expression (148) is proportional to the coefficient $B_{2m}$ we can suspect that the lowest mode usually is the most critical one. This also turns out to be the case for all the numerically computed equilibria.

For a circular cross-section the lowest odd mode corresponds to a translation and the lowest even mode to an expansion. If r is the radius of the cross-section of the plasma and $\varphi$ the angle of a circular coordinate system with the origin at the center of the cross-section the lowest modes of perturbation with displacement $\varepsilon$ give

$$x = r(1 + \frac{\varepsilon}{r} \cos \varphi) \cos \varphi$$

$$y = r(1 + \frac{\varepsilon}{r} \cos \varphi) \sin \varphi$$

and

$$x = r(1 + \frac{\varepsilon}{r} \sin \varphi) \cos \varphi$$

$$y = r(1 + \frac{\varepsilon}{r} \sin \varphi) \sin \varphi$$

for even and odd modes respectively. Squaring and adding gives to first order in $\varepsilon$

$$(x - \varepsilon)^2 + y^2 = r^2$$

for even modes and

$$x^2 + (y - \varepsilon)^2 = r^2$$

for odd modes.

That is the cross-section translates in horizontal or vertical direction without changing its shape.

If the cross-section differs from a circle these statements are no longer true, but as long as the difference is not too large we consider them to be approximately valid. If they are applicable in a special case might be a matter of judgement. We can thus consider stability against expansions and translations much more easily by applying the formalism of subsection 4.2 instead of the more laborious equations in subsection 4.1.

For a weak external field where the cross-section is a circle the first even mode is stable and the first odd mode is marginally stable according to (80 a) and (81 a). This is also verified by numerical computations of expressions (146)-(148).

For a stronger external magnetic field when the cross-section differs considerably from a circle, it turns out that $y_{max}$ occurs for a value of $\eta$ that is close to b. But the maximum displacement occurs for $\eta = b/2$. Figure 9 is the result from a numerical solution with the external magnetic field 0.425 of that on the plasma surface. We see that where the maximum displacement occurs for an odd perturbation of order 1 of the type (149) the boundary is nearly a straight line which means that $\partial s/\partial \eta$ is close to zero. That is the negative contribution from $\delta W_s$ to the change of potential energy is much less than it would be for a vertical displacement. Against axisymmetric modes of type (149) the plasma thus ought to be stable. This also turns out to be the case for the numerically computed example.

Stability of expansions and translations is not the same for the surface current model as it is for more realistic models as the mean-values in equation (80 a) and (80 b) are differently composed depending on how the current is distributed over the cross-section. If the cross-section of the plasma is small, however, the difference in the mean-values ought to be ignorable. We have therefore tested the stability for plasmas of small cross-section in an Extrap configuration. It turns out that most of the obtained equilibria are unstable, either against translations or expansions. All stable cases found were similar to the one showed in figure 10. If one of the conductors is situated straight over the plasma as in figure 11, it is unstable against expansions. If the conductors were situated one on each side of the plasma it is unstable against vertical translations. This can of course be explained by the fact that the current is flowing in opposite direction in the conductors and in the plasma, which therefore repel each other. No stable equilibrium was found for Extrap when the plasma surface was more close to the conductors, figures 12-18. But as both even and odd instabilities occur there might be stable equilibria also. It is evident that the equilibria that are unstable against even displacements can be stabilized by a vertical homogeneous magnetic

field. In one of the computed cases it was enough with a field-strength of seven percent of that on the plasma surface, see figure 19. If the homogeneous field is stronger, the aspect ratio became smaller but at the same time the conductors had to be placed more distant from the plasma, figure 20.

If the equilibrium is unstable against vertical displacements inequality (81 a) is violated. By applying an current flowing in the same direction as the plasma current, between the origin and the plasma, we add a vertical downwards pointing magnetic field to the external field inside the plasma. The strength of that field is decreasing with increasing x. That is

$$\frac{\partial B_{yIe3}}{\partial x} > 0$$

$I_{e3}$ refers to figure 21. In figure 21 such a case is shown. If the current is situated too close to the plasma the contribution to the total magnetic field from this current can be too large. Then inequality (80 a), which can be written

$$B_{ey} + x \frac{\partial B_{ey}}{\partial x} < 0$$

no longer is fulfilled. The equilibrium instead becomes unstable against expansions. Figure 22 shows such a case.

The Extrap scheme can be sligthly modified by rearranging the conductors in the way showed in figures 23-26.It turns out that this modification has a strong stabilizing effect on the perturbations we are studying here. In fact with the conductors arranged in that way, nearly all the computed equilibria were stable. If these equilibria are less stable against linear combinations of expansions and vertical translations is not investigated here. A couple of trials also was done with a homogeneous field and only one conductor figures 27 and 28. Both these equilibria turned out to be unstable, but at closer consideration it seems probable that stability could have been achieved if the conductor had been situated on the x-axis between the origin and the plasma and the direction of the current had been reversed, as in figure 21. Eventually the conductor could have been situated to the outer side of the plasma with the direction of the current unchanged.

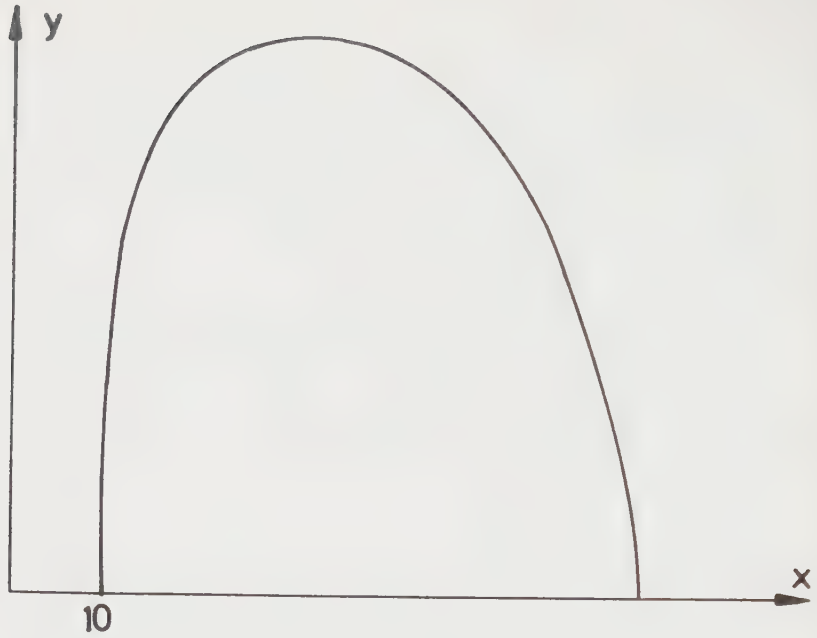



Figure 9. $I_p = 144$, $B_\infty = -0.425$

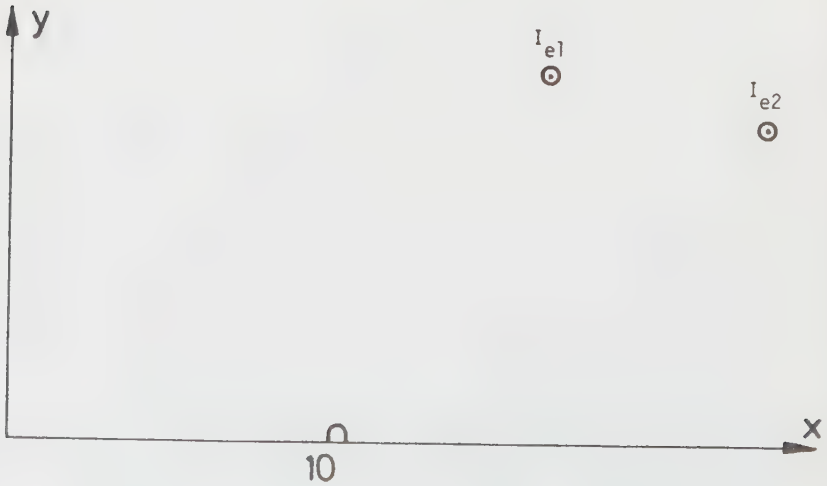



Figure 10. $I_{e1} = 0.737$, $I_{e2} = 1.20$, $I_e = 1.94$, $I_p = 2.93$, $I_e/I_p = 0.67$, p = 1/
Stable

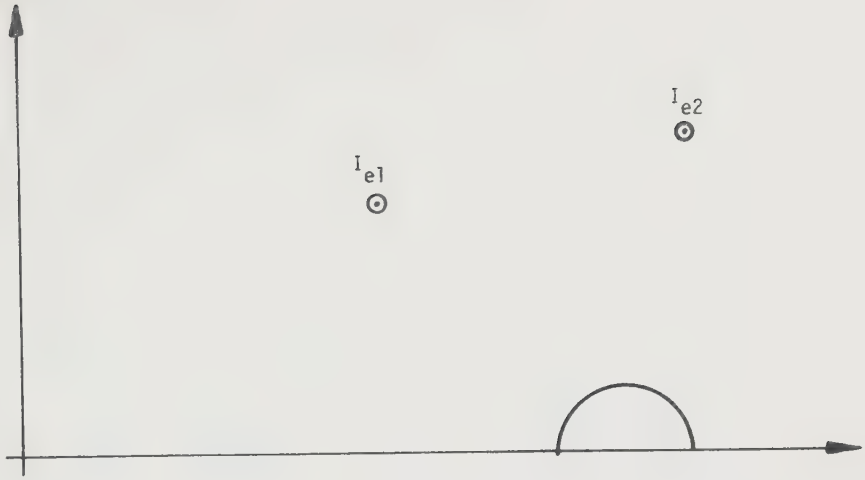



Figure 11. $I_{e1} = 15.9$, $I_{e2} = 25.8$, $I_e = 41.7$, $I_p = 7.1$, $I_e/I_p = 5.87$, $p = 1/2$
Unstable against the even mode.

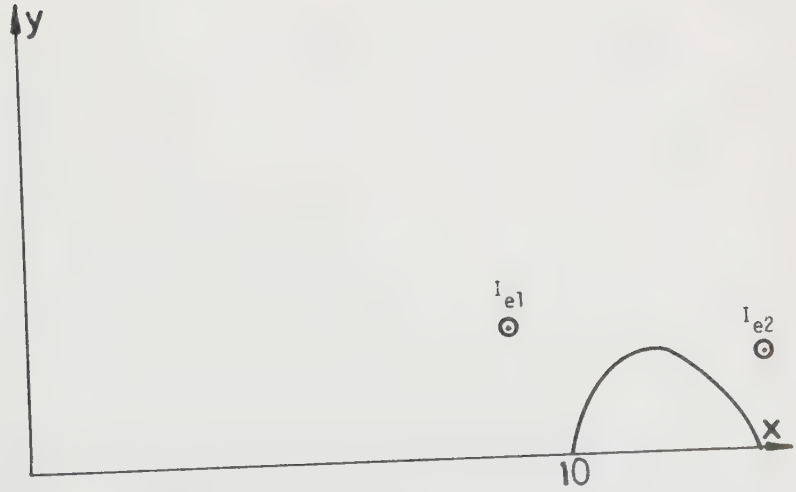



Figure 12. $I_{e1} = 3.093$, $I_{e2} = 3.093$, $I_e = 6.187$, $I_p = 11.05$, $I_e/I_p = 0.56$,
$p = 1/2$.
Unstable against the odd mode.

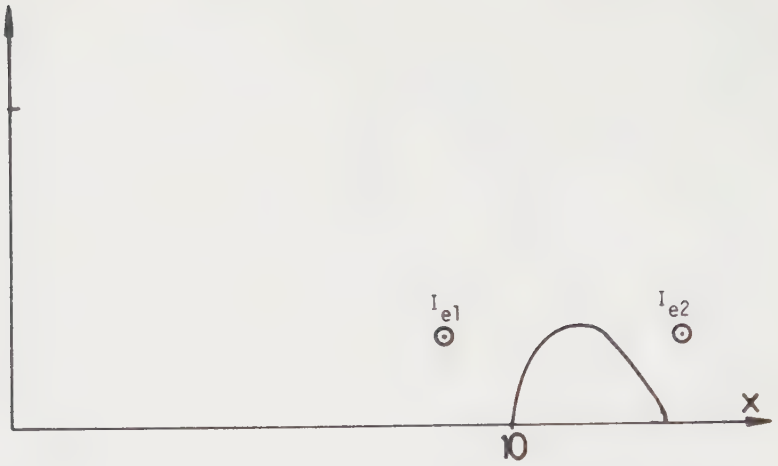



Figure 13. $I_{e1} = 3.975$, $I_{e2} = 3.975$, $I_e = 7.95$, $I_p = 11.13$, $I_e/I_p = 0.714$, $p = 1/2$.
Unstable against the odd mode.

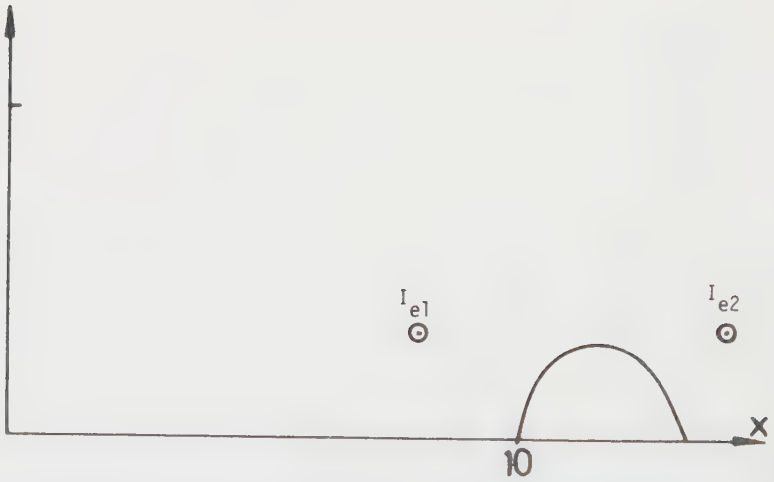



Figure 14. $I_e = 4.33$, $I_{e2} = 4.33$, $I_e = 8.67$, $I_p = 11.03$, $I_e/I_p = 0.79$, $p = 1/2$.
Unstable against the odd mode.

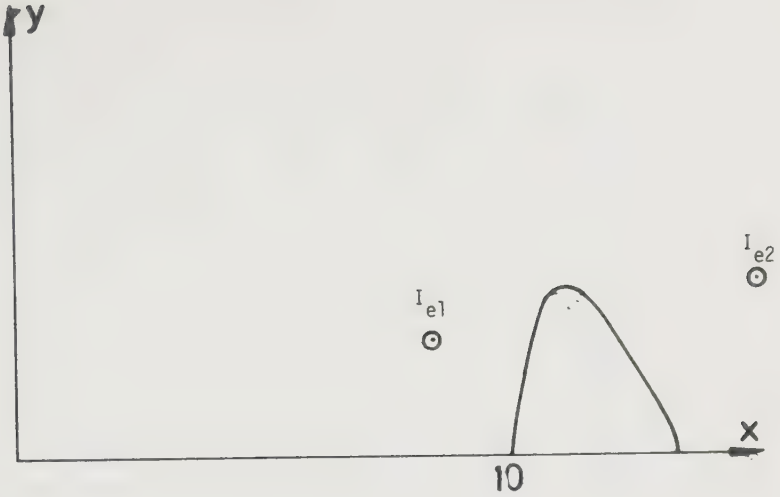



Figure 15. $I_{e1} = 17.17$, $I_{e2} = 17.17$, $I_e = 34.35$, $I_p = 16.34$, $I_e/I_p = 2.10$ $p = 1/2$.
Unstable against the odd mode.

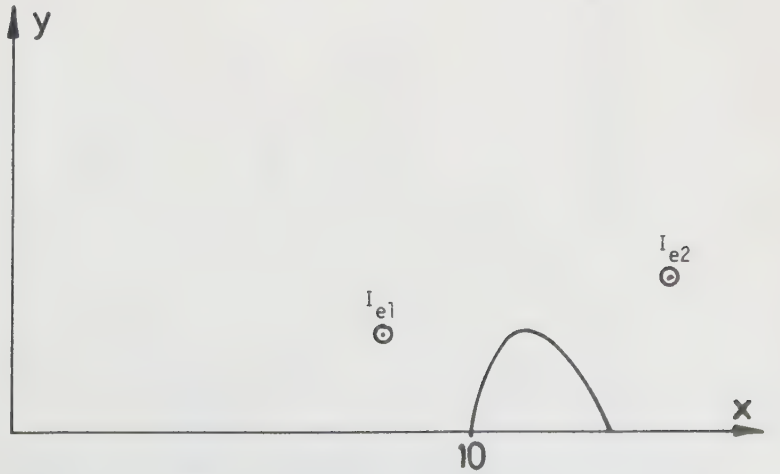



Figure 16. $I_{e1} = 13.60$, $I_{e2} = 13.60$, $I_e = 27.2$, $I_p = 11.16$, $I_e/I_p = 2.44$, $p = 1/2$.

Unstable against the odd mode.

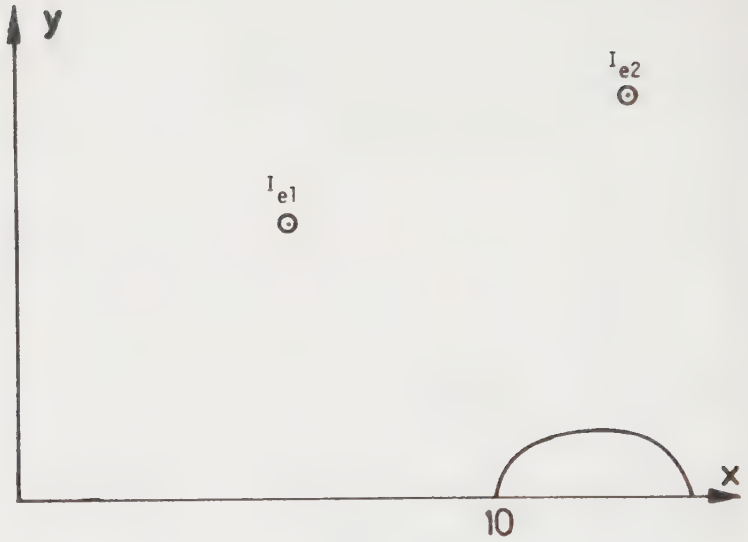



Figure 17. $I_{e1} = 17.87$, $I_{e2} = 21.44$, $I_e = 39.31$, $I_p = 11.04$, $I_e/I_p = 3.56$, $p = 1/2$.
Unstable against the even mode.

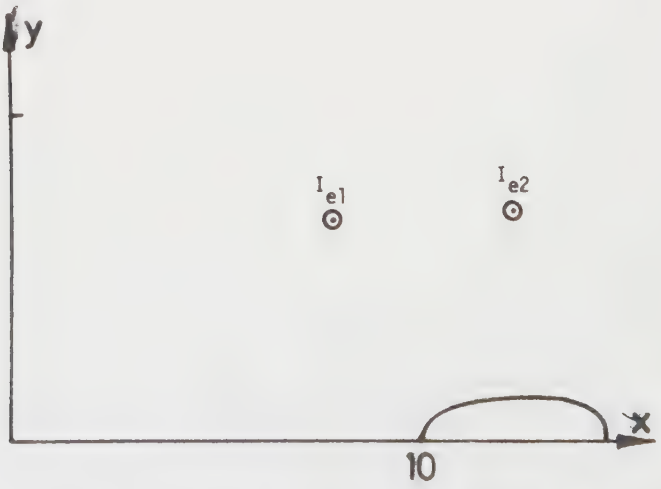



Figure 18. $I_{e1} = 11.63$, $I_{e2} = 30.49$, $I_p = 11.31$, $I_e/I_p = 2.70$, $p = 1/2$.
Unstable against the even mode.

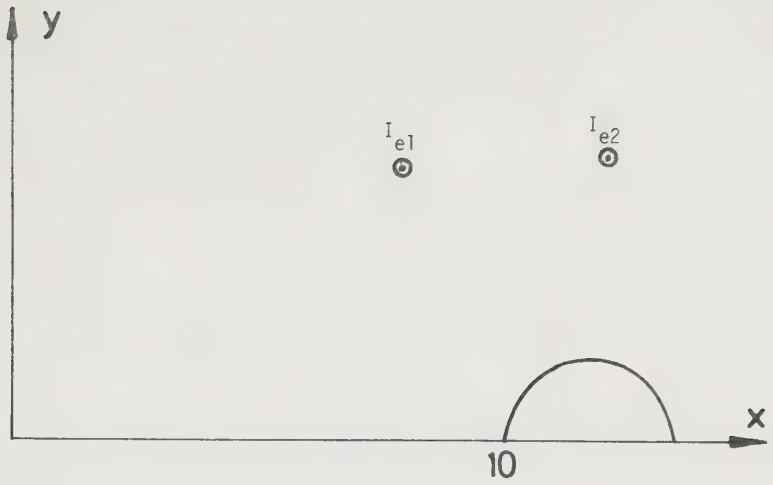



Figure 19. $I_{e1} = 0.735$, $I_{e2} = 1.19$, $I_e = 1.93$, $I_p = 10.97$, $I_e/I_p = 0.18$, $|B_\infty| = 0.07$.

Stable.

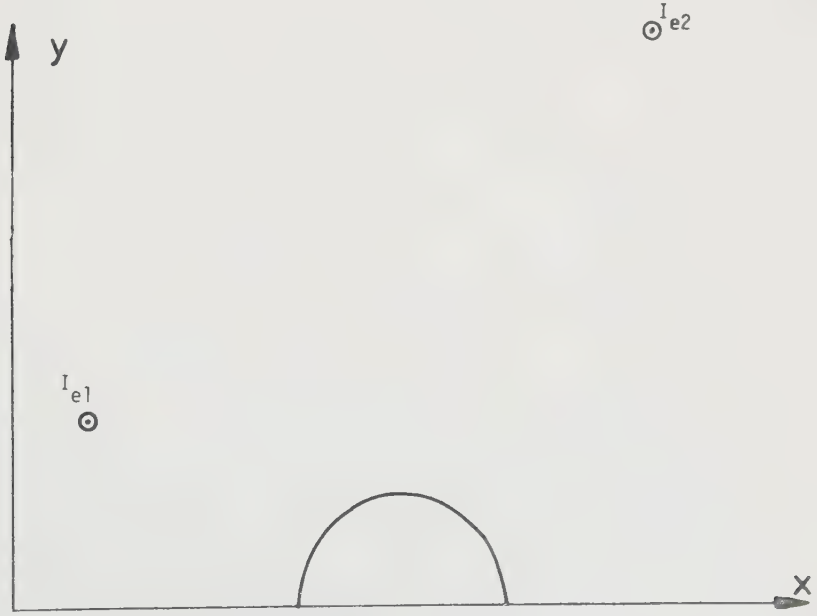



Figure 20. $I_{c1} = 10.97$, $I_{e2} = 5.476$, $I_e = 16.49$, $I_p = 23.85$, $I_e/I_p = 0.69$, $p = 1/2$, $B_\infty = 0.13$.

Stable

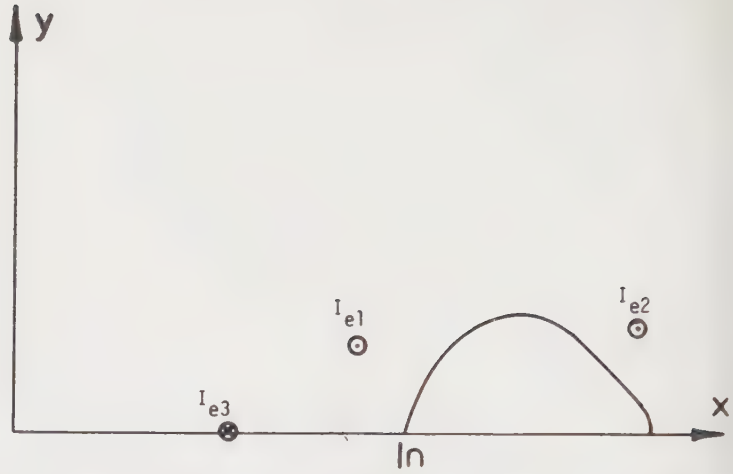



Figure 21. $I_{e1} = 3.256$, $I_{e2} = 3.256$, $I_{e3} = 6.511$, $I_p = 18.85$, $I_{e3}/I_p = -0.35$, $p = 1/2$.
Stable.

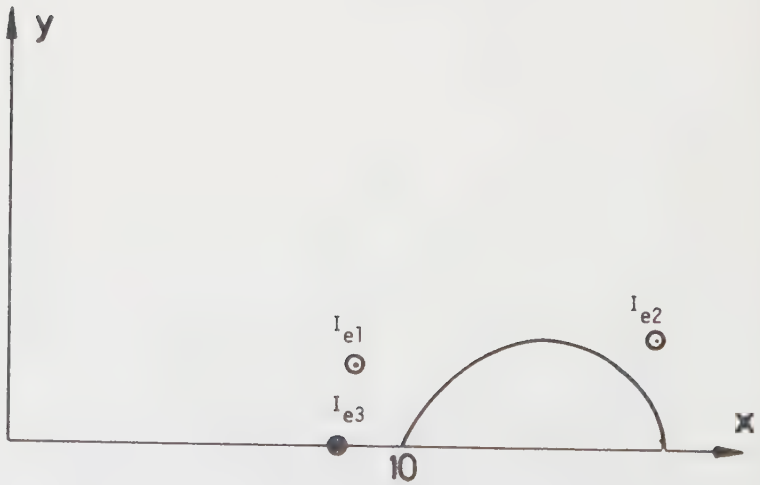



Figure 22. $I_{e1} = 1.76$, $I_{e2} = 1.76$, $I_{e3} = -3.52$, $I_p = 18.85$, $I_{e3}/I_p = -0.19$
$p = 1/2$

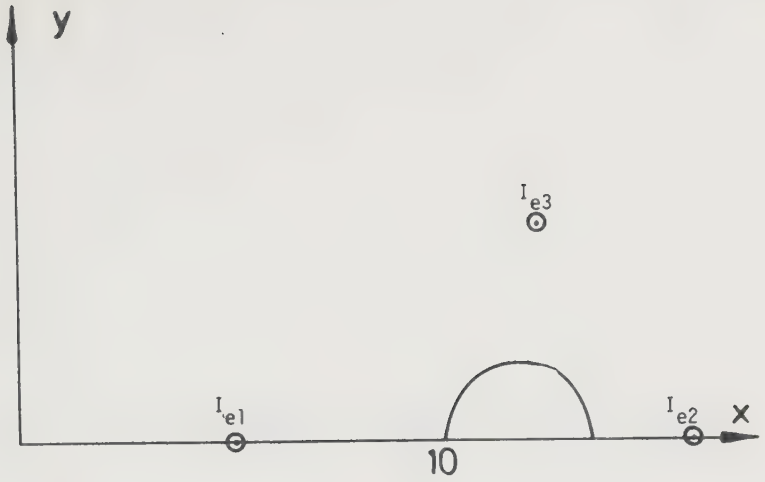



Figure 23. $I_{e1} = 3.061$, $I_{e2} = 3.061$, $I_{e3} = 3.061$, $I_e = 9.18$, $I_p = 10.97$, $I_e/I_p = 0.84$, $p = 1/2$.
Stable.

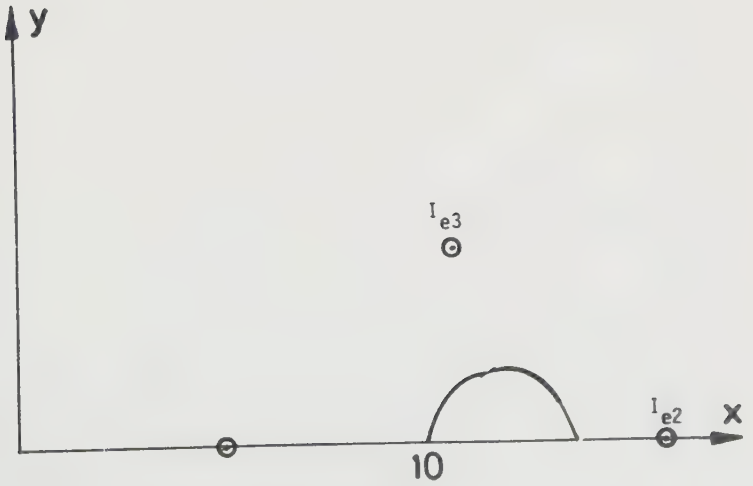



Figure 24. $I_{e1} = 5.16$, $I_{e2} = 5.16$, $I_{e3} = 5.16$, $I_e = 15.47$, $I_p = 26.74$, $I_e/I_p = 0.58$, $p = 1/2$.
Stable

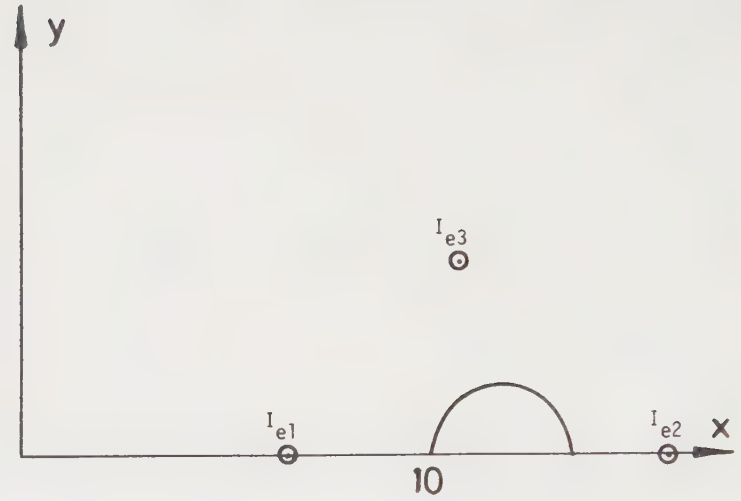



Figure 25. $I_{e1} = 6.84$, $I_{e2} = 6.84$, $I_{e3} = 6.84$, $I_e = 20.51$, $I_p = 10.98$, $I_e/I_p = 1.87$, $p = 1/2$.
Unstable against the odd mode.

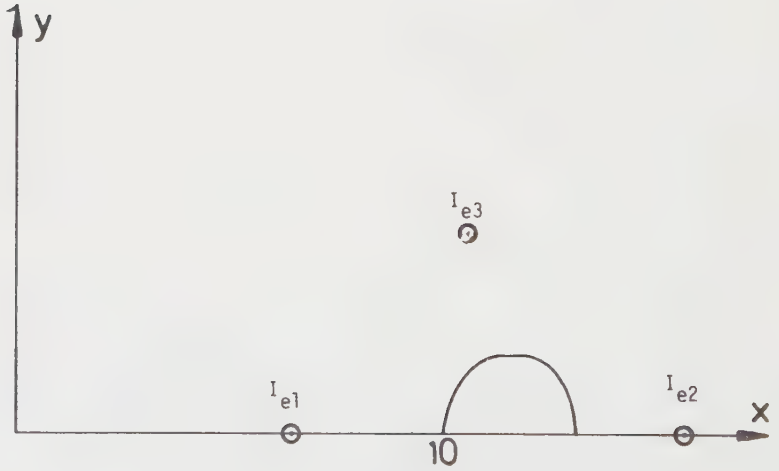



Figure 26. $I_{e1} = 6.63$, $I_{e2} = 6.63$, $I_{e3} = 3.31$, $I_e = 16.57$, $I_p = 11.01$, $I_e/I_p = 1.51$, $p = 1/2$.
Unstable against the odd mode.

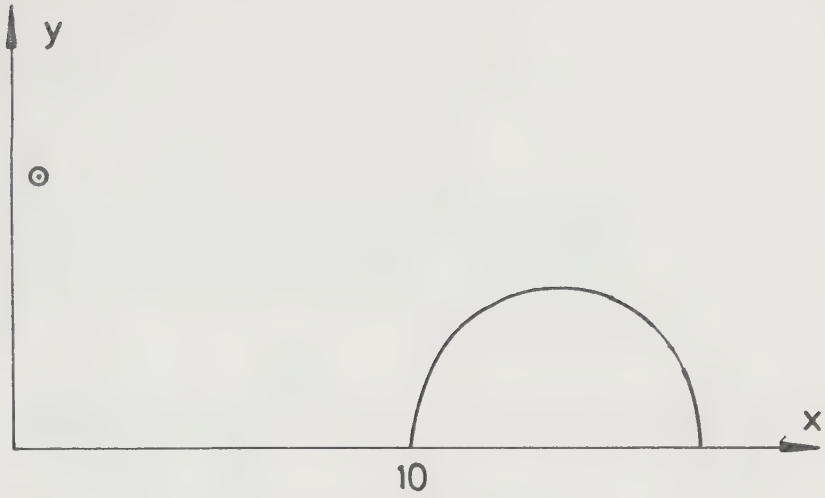



Figure 27. $I_c$ = 12,61, $I_p$ = 23.86, $I_c/I_p$ = 0.53, $B_\infty$ = - 0.15
Unstable against the odde model

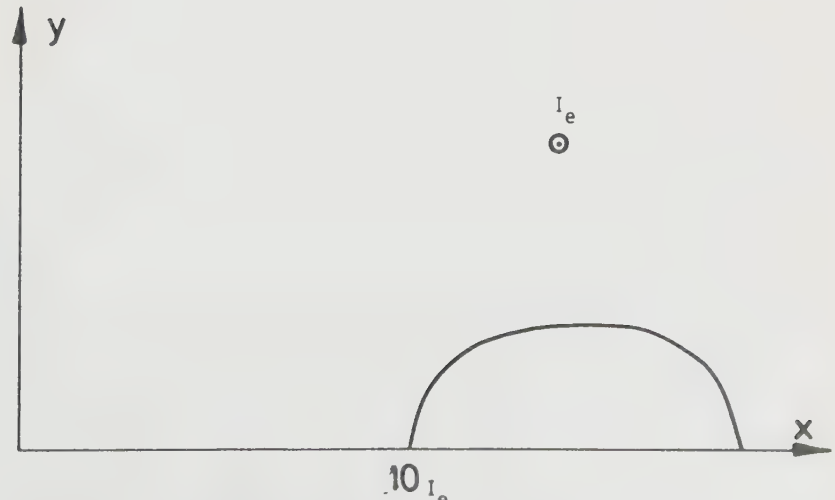



Figure 28. $I_c$ = 8.55, $I_p$ = 23.87, $\frac{I_e}{I_p}$ = 0.36, $B_\infty$ = - 0.1.
Unstable against the even mode.

## 6. REVIEW AND VIEWS

We have started from the idea to first compute the shape of the plasma cross-section and the magnetic field outside the plasma, instead of walking the usual path and regard the cross-section and the current distribution to be given, and compute an equilibrium with regard to the magnetic field and pressure.

From our starting point we have succeeded to work out a method which solved the problem outside the plasma when the sources of the external magnetic field were given. We have also derived an equation that determines the magnetic field, current and pressure inside the plasma, and boundary conditions for that equation have been decided. No attempt to solve that equation has been made.

The advantage of our method is that it is possible to determine equilibria starting from a given type of external sources rather than obtain the type of sources as a final result of the computation. The price we had to pay for the obtained result was that we had to renounce the complexity of the topology and,instead of a real torus,model one by means of two straight plasma columns. The decisive gain of this model is that we can decide a simple domain in an auxiliary plane that we can take as the vacuum region in that plane, and then find a transformation that transforms the boundary back to the z-plane properly. A condition for this to be possible is that we can find a transformation that preserves Laplace's equation.

Some promising preliminary considerations have been made to find transformations that make it possible to treat the equations for a real toroidal plasma in a similar manner but the work has not yet progressed far enough to make it possible to predict if there will be any final success.

The method we have worked out is of course not limited to cases similar to the one we have treated. It would for example be possible to treat the case where the plasma is shielded by a conducting shell and surrounded by some irregularly distributed current-carying conductors inside the shell. This configuration can be mapped onto a vertical infinite strip. The situation is sketched in figure 29. If there are four symmetrically spaced conductors, the mapping of figure 30 would be more simple.

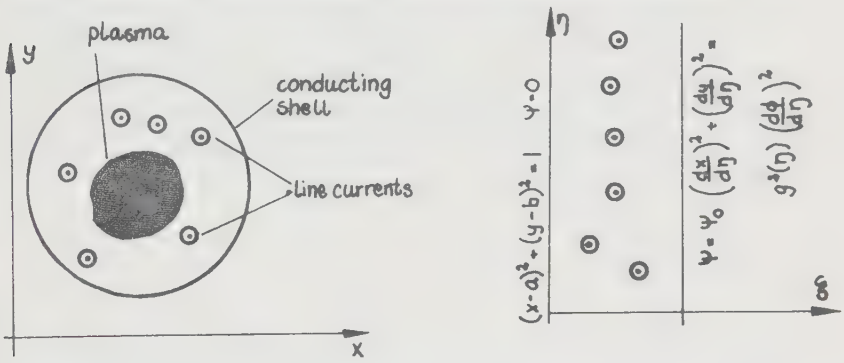



Figure 29. Possible mapping for solving the problem of a plasma surrounded by line-currents and a conducting shell. The shell can of course be substituted by infinity.

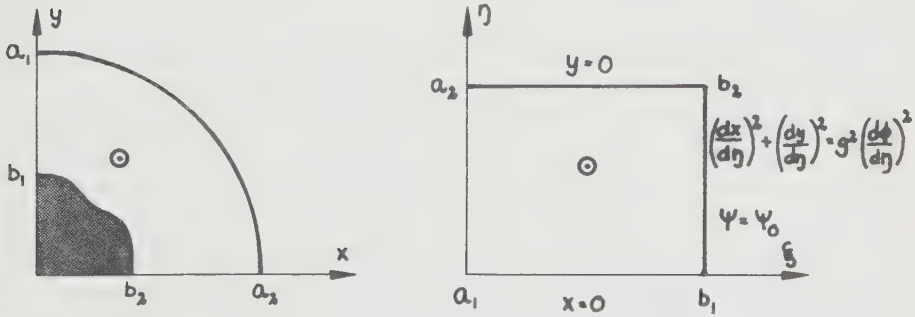



Figure 30. For four symmetrically situated conductors this mapping may be more simple.

It also turned out that the method simplifies the stability analysis, especially for axisymmetric perturbations. This depends on that the potential energy in the vacuum region is a solution of Laplace's equation.

and for two dimensions this equation is invariant under a conformal mapping. For the kink modes the mapping gave us a more complicated equation as the potential that minimized the energy-change in that case also depends on z and thus is not a solution to the two dimensional Laplace equation.

In the stability investigation we have used the energy-principle, but it would also be possible to apply the mapping for a normal mode analysis as also the boundary condition (99) can be expressed in terms of the mapping. For axisymmetric modes and the surface current model simple analysis shows that the boundary condition of equation (102) is given by expression (130) where $\psi$ is the solution of Laplace equation whith the mixed boundary condition

$$\frac{\partial \psi}{\partial \xi} + \frac{\partial s}{\partial \eta} \psi = 0 \qquad \xi = a,\ 0 < \eta < b$$

When we applied the energy principle we paid no attention to the result of ref. [17]. In this reference is shown that under special conditions, unbounded solutions to the linearized MHD-equations can occur, even when the energy-principle predicts the plasma to be stable.

We analyzed only a very special type of instability, namely axisymmetric modes and for them only simple even and odd harmonics, but we have shown in the preceding section that we can always find a linear combination of such harmonics for which the equilibria we have investigated are unstable.

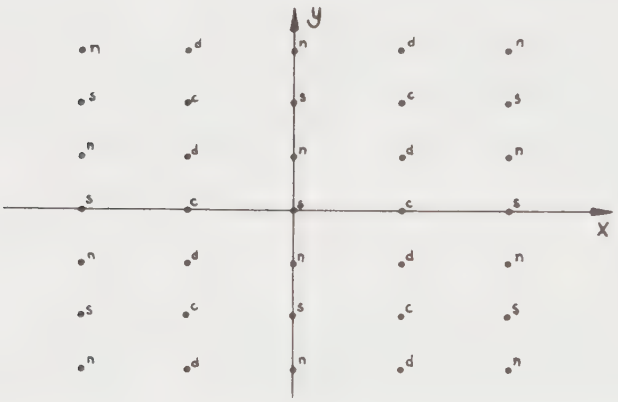
Finally, we want to point out that the most interesting analyses remain to be done, namely to solve equation (26) with boundary condtion (47) and for the obtained equilibrium solve equation (25) to determine the internal pressure,magnetic field and current density. We then have an equilibrium to which we can apply more general stability theories, for example those found in references [16, 18 - 20].

ACKNOWLEDGEMENTS

I wish to thank professor Erik T Karlson, who introduced me to this subject and proposed the original problem, for his constant support through this work, Dr Horst Stiewe for elucidating and valuable discussions at the initial stage of the work, and Dr Hans Olofsson for invaluable help with the numerical calculations.
I am also very greatful to mrs Margareta Hallenberg, who skilfully typed the manuscript and patiently interpreted my hand-writing, and to Efva Pålholm, Ingrid Strandh and Kerstin Hedenstedt,who drew the figures.

APPENDIX A

Cover the plane with a rectangular point pattern with sides K and $K^1$ in x resp y direction. Let one of the points be at the origin (see fig.)




Name the point at the origin and every two points in horizontal and vertical direction s so that s appears with periodicity $2K^1$ in vertical and 2K in horizontal direction. Name the remaining points c, d or n with the same periodicity in both directions as shown in the figure.
The elliptic function

p q (z/m)

is an analytic function with a zero at every point named p and a pole in every point named q in the figure above. The elliptic functions are normalized so that

$$\lim_{z \to 0} p\ q\ (z/m) = \begin{cases} z \text{ if p is s} \\ \frac{1}{z} \text{ if q is s} \\ 1 \text{ if p, q is c, d or n} \end{cases}$$

m is called the parameter of the elliptic functions and is related to the quarter periods K and K' by

$$K(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-m \sin^2\varphi}}$$

$$K^1(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-(m-1)\sin^2\varphi}}$$

It is common to introduce the complementary parameter m' m + m' = 1.

The functions that fit the definition above of elliptic functions can be written.

$$z = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-m^2\sin^2\varphi'}} \, d\varphi'$$

$$sn(z|m) = \sin \varphi$$

$$cn(2|m) = \cos \varphi$$

$$dn(z,m) = \sqrt{1-m \sin^2\varphi}$$

From these three functions the remaining elliptic functions can easily be computed by

$$pq(z|m) = \frac{pn(z|m)}{qn(z|m)}$$

The figure gives easily the periodicity of the elliptic functions:

sn has periods 4K, 2iK', 4(K+iK')

cn has periods 4K, 4iK', 2(K+iK')

dn has periods 2K, 4iK', 4(K+iK')

Translation a quarter period is easily computed by inspection of the figure. The change of normalization must, however, be remembered.

For example we have

$$\lim_{z \to C} dn(z|m) = \sqrt{1-m} = \sqrt{m'}$$

which gives

$$dn(z+K|m) = \sqrt{m'}\ nd(z+K|m)$$

The elliptic functions of imaginary arguments can also easily be found by inspection of the figure. We have

$$sn(iz|m) = isc(z|m')$$

$$cn(iz|m) = nc(z|m')$$

$$dn(iz|m) = dc(z|m')$$

The properties of the elliptic functions which are described above are exactly what is needed to follow this work. For more details is referred to [ 9] and [10].

APPENDIX B

At the origin and at infinity the function W assumes the values

$$W_o = i \frac{\frac{c}{b} K' - \frac{\pi}{b} \Sigma\, n\, A_n - \frac{\phi_o}{c}}{\exp\left[\sum_{n=o}^{\infty} B_n\right]} \qquad (B\ 1)$$

and

$$W_\infty = i \frac{c}{b} \frac{\pi^2}{4\sqrt{m}\, K' \exp\left[\Sigma(-1)^n B_n\right]} \qquad (B\ 2)$$

respectively.

From expression (B2) we conclude that a small magnetic field at infinity corresponds to a dominant zero order term in the Fourier sum of equation (67). That is, the third term of the right hand side of this equation is small compared to the sum of the first and the second terms and the last term is nearly equal to $\ln(\phi_o/c)$. This occurs when a >> b as seen from expression (60). In this limit equations (67) and (54) can be expanded in Taylor series. With

$$\alpha = \text{tgh}\, \frac{a}{2}$$

$\alpha$ is close to unity, we can therefore write

$$\ln\left\{\cos^2 \frac{\pi\eta}{2b} + \alpha^2 \sin^2 \frac{\pi\eta}{2b}\right\} = -(1-\alpha)(1-\cos\frac{\pi\eta}{b}) + O\left[(1-\alpha)^2\right]$$

$$\frac{1}{\sinh \frac{\pi a}{b}} = (1-\alpha)\left[1 + \frac{1}{2}(1-\alpha) + \frac{1}{2}(1-\alpha)^2\right] + O\left[(1-\alpha)^4\right]$$

$$\frac{1}{\cosh \frac{\pi a}{b}} = (1-\alpha)\left[1 + \frac{1}{2}(1-\alpha)\right] + O\left[(1-\alpha)^4\right]$$

$$\text{tgh}\,\frac{\pi a}{b} = 1 - \frac{1}{2}(1-\alpha)^2 + 0\left[(1-\alpha)^3\right]$$

$A_n \cosh \frac{n-a}{b}$ is therefore of order $(1-\alpha)^n$. The last term of the right side of equation (67) thus can be written

$$\ln \frac{\phi_o}{c} + \frac{c\pi}{\phi_o} A_1 \cosh \frac{\pi a}{b} \cos \frac{\pi\eta}{b} + \frac{c\pi}{\phi_o} A_e \cosh \frac{2\pi a}{b} \cos \frac{2\pi\eta}{b} - \frac{\pi^2 c^2}{4\phi_o^2} A_1^2$$

$$\cosh^2 \frac{\pi a}{b} \left(1 + \cos \frac{2\pi\eta}{b}\right)$$

The first few coefficients of $B_n$ are therefore

$$B_o = 2 \ln \left\{\cosh \frac{\pi a}{2b}\right\} + \ln \frac{\phi_o}{b} - (1-\alpha) - \frac{1}{4}(1-\alpha)^2 - \frac{\pi^2 c^2}{4\phi_o^2} A_1^2 \cosh^2 \frac{\pi a}{b} \qquad \text{(B 3)}$$

$$B_1 \cosh \frac{\pi a}{b} = (1-\alpha) + \frac{c\pi}{\phi_o} A_1 \cosh \frac{\pi a}{b} + \frac{1}{2}(1-\alpha)^2 \qquad \text{(B 4)}$$

$$B_2 \cosh \frac{\pi a}{b} = 0\left[\frac{\pi c}{\phi_o} A_2 \cosh \frac{2\pi a}{b} - \frac{\pi c^2}{4_o} A_1^2 \cosh \frac{2\pi a}{b}\right](1-\alpha)^2 \qquad \text{(B 5)}$$

exp (r) and s can also be expanded in powers of $(1-\alpha)$. To the first order the result is

$$c^{r(a+in)} = \frac{\phi_o}{b}\left\{1 - (1-\alpha)\, 2\, \frac{\pi c}{\phi_o} \cos \frac{\pi\eta}{b}\right\} \qquad \text{(B 6)}$$

$$s(a + i\eta) = -\frac{\pi\eta}{b} + (1-\alpha)\, 2\left(1 - \frac{\pi c}{\phi_o}\right) \sin \frac{\pi\eta}{b} \qquad \text{(B 7)}$$

Inserted into equation (54) this gives

$$\int_o^b \left[1 - (1-\alpha)\, 2\, \frac{\pi c}{\phi_o} \cos \frac{\pi\eta}{b}\right]\left[\cos \frac{\pi\eta}{b} +\right.$$

$$\left. + (1-\alpha)\, 2\left(1 - \frac{\pi c}{\phi_o}\right) \sin^2 \frac{\pi\eta}{b}\right] d\eta = 0$$

which shall be solved for $\phi_o/c$. The solution is

$$\frac{\phi_o}{c} = 2\pi \qquad \text{(B 8)}$$

Expression (B3) and (B4) can now be written

$$B_0 = \ln \frac{1}{1-\alpha^2} + \ln \frac{2\pi c}{b} - \left[(1 - \alpha)\right] \qquad \text{(B 9)}$$

$$B_1 = 0 \quad \left[(1 - \alpha)^4\right]$$

$$B_2 = \frac{1}{2} \quad (1 - \alpha)^2 \quad \frac{1}{\cosh \frac{2\pi a}{b}}$$

In the considered limit we also have that

$$m \approx 1$$

and

$$K \approx \frac{\pi}{2}$$

so that

$$K' \sqrt{m} = \frac{\pi}{2} \left(1 - \frac{m_1}{4m}\right) + 0 \left[\left(\frac{m_1}{m}\right)^2\right]$$

Inserting this and (B9) into expression (B2) gives

$$1 - \alpha = 2 \, |W_\infty| (1 - |W_\infty|) \qquad \text{(B 10)}$$

which together with (B6) and (B7) allows us to write

$$\exp(a + i\,\eta) = \frac{\phi_o}{b} \left\{1 - 2 \, |W_\infty| \cos \frac{\pi\eta}{b}\right\} \qquad \text{(B 11)}$$

$$\cos s(a + i\,\eta) = \cos \frac{\pi\eta}{b} + 2 \, |W_\infty| \sin \frac{2\pi\eta}{b} \qquad \text{(B 12)}$$

$$\sin s(a + i\,\eta) = - \sin \frac{\pi\eta}{b} + 2 \, |W_\infty| \cos \frac{\pi\eta}{b} \sin \frac{\pi\eta}{b} \qquad \text{(B 13)}$$

$$\exp\,[r(\xi)] = \frac{\exp(B_o)}{\cosh \frac{\pi\xi}{2b}} \qquad \text{(B 14)}$$

Expressions (55) and (56) can now be computed

$$x_1 = \int_o^a \exp\,[r(\xi)]\,d\xi = \exp\,(B_o)\,\frac{2b}{\pi}\,\mathrm{tgh}\,\frac{\pi a}{2b} =$$

$$= \frac{\phi_o}{2\pi|W_\infty|}\left(1 - 2\,|W_\infty| + 2\,|W_\infty|^2\right) \qquad \text{(B 15)}$$

$$z = x_1 + \frac{c_o}{b}\int_o^\eta \left\{\left(1 - 2\,|W_\infty|\,\cos\frac{\pi\eta'}{b}\right)\right.$$

$$\left.\left[\sin\frac{\pi\eta'}{b} - 2\,|W_\infty|\,\cos\frac{\pi\eta'}{b}\,\sin\frac{\pi\eta'}{b} + i\left(\cos\frac{\pi\eta'}{b} + 2\,|W_\infty|\,\sin\frac{\pi\eta'}{b}\right)\right]\right\}\,d\eta$$

$$= \frac{\phi_o}{2\pi|W_\infty|}\left\{1 - 2\,|W_\infty|\left(\cos\frac{\pi\eta}{b} - i\,\sin\frac{\pi\eta}{b}\right)\right.$$

$$\left. + 2\,|W_\infty|^2\left(\cos\frac{2\pi\eta}{b} - i\,\sin\frac{\pi\eta}{b}\right)\right\}$$

In the expression for the flux

$$\psi_o = c\left[\frac{\pi}{2\sqrt{m}\,x'} - \frac{\phi_o a}{cb}\right]$$

insert $\psi_o = -1$ which is the value used in ref [7]. That gives

$$\phi_o = \frac{\pi}{\ln\frac{1}{|W_\infty|}}$$

which inserted in the expression for z gives

$$z = \frac{\frac{1}{|W_\infty|}}{2 \ln \frac{1}{|W_\infty|}} \left\{ 1 - 2 |W_\infty| \left( \cos \frac{\pi\eta}{b} - i \sin \frac{\pi\eta}{b} \right) \right.$$

$$\left. + 2 |W_\infty|^2 \left( \cos \frac{\pi\eta}{b} - i \sin \frac{\pi\eta}{b} \right) \right\} \qquad \text{(B 16)}$$

To the first order in the asymptotic magnetic field this is a circle with radius $-1/\ln |W_\infty|$ and the center in

$$z = \frac{\frac{1}{|W_\infty|}}{2 \ln \frac{1}{|W_\infty|}}$$

This expression differs with a term $-1/\ln |W_\infty|$ from the result in ref. [7]. If we compute $x_1$ to zero order we get

$$x_1 = \frac{\frac{1}{|W_\infty|}}{2 \ln \frac{1}{|W_\infty|}}$$

and if we then add the radius we get the result that the center of the circle is in

$$z = \frac{\frac{1}{W_\infty}}{2 \ln \frac{1}{|W_\infty|}} (1 + 2 |W_\infty|)$$

which agrees exactly with the result of ref. [7].

To first order in the asymptotic magnetic field the ratio of $W_0$ to $W$ is

$$\frac{W_0}{W_\infty} = \frac{\frac{\pi}{2} - 2\pi}{\frac{\pi^2}{2\pi}} = -3$$

which also agrees with the result of ref. [7].

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